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ABSTRACT

Part I of this booklet contains a brief report of the procedures and the philosophy of the SMSG writing teams for each of the following areas: junior high school mathematics; algebra; geometry; geometry with coordinates; eleventh-grade mathematics; twelfth-grade mathematics; and mathematics for grades 4, 5, and 6. Part II includes a criticism of SMSG text material and a review of the SMSG First Course in Algebra. (DT)

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PREFACE

The procedure used by SMSG in the preparation of its texts--large writing teams working intensively for a relatively short time--is not a common one. It seems desirable, therefore, to present a record of just how each writing team went about its task, what kinds of problems it faced, and how it reached decisions.

In addition, while the philosophy of each writing team with respect to its part of the mathematics curriculum is implicit in the text produced by the writing team, it seems important to have more concise statements of the philosophies. In particular, it has always been the hope of SMSG that the authors of commercial textbooks will make extensive use of SMSG materials. An outline of the underlying philosophy of each SMSG text may well be of use to such authors.

Consequently, after each SMSG text was completed, a representative of the writing team was asked to prepare, in consultation with his colleagues, a brief report of the procedures and of the philosophy of his writing team.

Part I of this booklet is devoted to these reports.

While expressions of opinion, pro and con, concerning the SMSG texts have been put forth in great quantity, substantive discussions of the materials in these texts have been very scarce. One example consists of an article "Sampling a Mathematical Sample Text" by A. Wittenberg and a response, "Some Reflections on the Teaching of Area and Volume" by E. E. Moise.*

Two other constructively critical statements on SMSG texts have been prepared for the SMSG Advisory Board and SMSG writing teams. The Board agreed that these two statements would be as interesting and as useful as the statements mentioned above and they are, therefore, included in this booklet as Part II.

The first of these statements was prepared by N. E. Steenrod after a careful inspection of the preliminary editions of the SMSG texts for grades 7-12. The second was prepared more recently by P. D. Lax after a careful inspection of the text "First Course in Algebra.

It is hoped that publication of these two statements will stimulate further substantive criticisms and discussion of SMSG texts.

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* American Mathematical Monthly, vol. 70 (1963) pp. 452-459 and pp. 459-466.

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THE JUNIOR HIGH SCHOOL MATHEMATICS WORK OF THE SCHOOL MATHEMATICS STUDY GROUP

by

John Mayor

Selection of Content and Organization

The beliefs that mathematics is fascinating to many persons because of opportunities for creation and discovery as well as for utility, and that mathematics can and should be studied with enjoyment, were basic considerations in the plans for the first experimental units for grades seven and eight which were written at Yale University in the summer of 1958. The 1958 and later writers recognized, first, that junior high school students can formulate mathematical questions and conjectures which the students can test and perhaps settle and, second, that students can develop systematic attacks on mathematical problems whether or not the problems have routine or immediately determinable solutions. The teaching experience with the units during the next years provide overwhelming confirmation that the assumptions on which the choices of content were made during the three years of the work on junior high school mathematics were correct. The hope of the writers that seventh and eighth grade students could study more sophisticated mathematics with success and enjoyment has also been clearly justified.

Traditional mathematics for grades 7 and 8 might properly be described as general mathematics, including topics from arithmetic, geometry, and sometimes trigonometry and algebra. In the traditional courses there are great emphases on the social applications, with the social applications being interpreted narrowly to mean applications to problems of business. There have been practically no applications to science. Far too much of the time in courses at this level has been devoted to review of topics learned in the first six grades, and there has been no organization which could be said to take advantage of structure in mathematics.

During the summer of 1958 no attempt was made to develop a unified course. Rather, the small writing team tried to write experimental units on topics, not then appearing in the seventh and eighth grade texts, which they felt could be taught with interest and profit at this level and as appropriate replacement for some of the traditional content. During the four-week session in 1958, however, and particularly during the meetings of the Panel on Junior High School Mathematics in the ensuing academic year of 1958-59, major decisions were made about the nature of courses for grades 7 and 8. The idea of a course in algebra or a course in geometry, as such, was rejected in favor of courses for grades 7 and 8 in which there would be as much emphasis as possible on mathematical structure and in which materials would be selected from arithmetic, algebra, geometry, and trignometry. It was agreed that the work in social mathematics was greatly overemphasized in the seventh and eighth grades and, if this were to be included at all, it should come as applications of mathematical ideas and procedures. It was further agreed that there would be no review or drill sections on topics previously studied, but that maintenance of skills would be provided for in new settings, such as the system of rational numbers or metric and non-metric relations in geometry. From the beginning and throughout the material mathematical ideas have been associated with their applications, but the applications have been secondary, not an end in themselves. It is fruitless to train students in applications that will soon be outmoded. They must have the basic knowledge from which to make whatever applications are appropriate to their time.

The group also accepted the hypothesis that at this level experience with and appreciation of abstract concepts, the role of definition, the development of precise vocabulary and thought, experimentation and proof, were essential and appropriate. Experience in teaching the materials and the sample texts has also confirmed that this is a sound hypothesis.

There has been discussion by the Panel and by the writing teams as to whether the mathematics of grades 7 and 8 should be sequential. For example, would it be necessary always to teach Chapter 5 before Chapter 6? Although there has been some support for this point of view, this is not characteristic of the two courses in all instances. Particularly in the course for grade 8, considerable variation could be followed in the teaching order of the chapters. The decision for the inter-weaving of ideas from arithmetic, algebra, and geometry and for the selection of some materials which are not necessarily dependent on preceding chapters has resulted from the belief of the writers that twelve- and thirteen-year-old children enjoy some change of pace and change of topic; that is, after three chapters on ideas from arithmetic, a chapter on geometry might be appropriate, or that an introduction to probability and permutations and combinations is highly important in grades 7 and 8, even though these topics do not necessarily fit into a sequential arrangement.

Other decisions have influenced the general development of the materials, both of which might be described in terms of interest in emphasizing structure in mathematics. In algebra and arithmetic, the structure is emphasized by dealing in successive chapters with certain systems of numbers, such as the system of whole numbers, the system of rational numbers, the real number system, and finite mathematical systems. The common structure of all these is emphasized and each is used to reinforce the other. In the geometry, after the experience of the first year, the idea has been to begin the study with non-metric notions to emphasize first this part of the structure. This is followed by a careful analysis of the idea of measurement which is used as a basis for an inductive and informal approach to the acceptance by the children of certain fundamental properties of geometrical figures. From these properties, others are found by informal deduction.

Stages in the Development of the Courses

During the first summer's work in writing experimental units, there was close cooperation between the University of Maryland Mathematics Project (Junior High School) and the School Mathematics Study Group. As a result, some of the chapters in the seventh grade books of the University of Maryland Mathematics Project and School Mathematics Study Group are quite similar except for editorial changes. Some of these chapters were originally written by the group working at Yale University and others by the group working at the University of Maryland. Two special characteristics of the Maryland courses are: (1) most of the material is developed from the point of view of number systems--much more than in the SMSG development--and, (2) the grade 7 and 8 courses are written so that a student finishing these two courses will have completed a first year of algebra.

The writing teams have rejected both of these emphases for the School Mathematics Study Group material. Although there was interest in mathematical systems as such, it was felt that this particular idea could be overemphasized at the loss of some important related topics, applications, and materials from geometry. Secondly, because the School Mathematics Study Group had a substantial first-year course in algebra, it was believed to be definitely undesirable to duplicate this material in the seventh and eighth grade courses. Furthermore, since there was difficulty in limiting the amount of materials, it seemed even more desirable to delay the formal study of algebra until the ninth grade or until the offering of a first-year course in algebra.

During the first summer at Yale University, fourteen experimental units were written. Ten Centers were organized for the tryout of materials. In each Center there was a mathematician who worked regularly with the teachers, giving them background mathematics desirable for teaching the new materials and, where possible, assisting them with the teaching. The Centers were asked to report regularly on experiences.

In November of 1958 a conference of all teachers in the various Centers was held in Washington to talk about the experimental units, the point of view, and the teaching problems. The Panel for Grades 7 and 8 met a number of times that year studying the teacher reports and making plans for the development of the seventh grade course during the summer of 1959 at the University of Michigan.

In the spring of 1959 the Panel for Grades 7 and 8 outlined a course for seventh grade and prepared a suggested outline of units, some of which were to be written and tried out in the eighth grade. During the eight weeks' work at the University of Michigan in the summer of 1959, in which were involved some 15 mathematicians and 15 teachers, the course for grade 7 was prepared in experimental form, and the course for grade 8 was written as sample units combined into a book which might be taught as a regular text for grade 8.

In the fall of 1959 teachers in the Centers for grades 7 and 8 joined teachers in other Centers in a conference in Chicago at which the purpose and scope of the School Mathematics Study Group were carefully reviewed.

The Centers were continued for the academic year 1959-1960 and teacher reports were received from over 100 teachers on their experience in teaching both the seventh and eighth grade materials. In general, the teacher reports have been enthusiastic. Teachers did not fail to point out certain sections which were too difficult or which, from their point of view, were poorly written; but in general, they accepted all of the materials prepared for grades 7 and 8. There were a few suggestions for additional materials. Perhaps the suggestion for additional materials occurring most frequently was that there should be more work in algebra, particularly in operations with negative numbers and in the solution of equations. Again, the writing teams did not take the advice or follow the request of the teachers in this respect, largely because of the existence of, and the quality of, the first-year algebra course in the School Mathematics Study Group series.

During eight weeks at Stanford University in the summer of 1960, twenty mathematicians and teachers revised the seventh and eighth grade courses and organized the units which had been tried out in grade 8 into a course for careful tryout for another year. A number of changes in the grade 8 course were made at the suggestion of other writing teams. More work on constructions, including instruments in addition to ruler and compass; and introduction to sketching three-dimensional figures; and a treatment of symmetry were added at the request of the grade 10 team. The chapters on algebra were somewhat simplified and the language was made consistent with that of the algebra courses to follow. Frequent conferences with other writing teams served to make the grade 7 and 8 materials part of a more carefully planned mathematics program for grades 4 through 12. It is cumulative of what went before and exploratory of what is to come.

During the school year 1960-61 the materials for grade 8 were tried out as a sample textbook and, in a four-week session during the summer of 1961 at Yale University, these were written in a somewhat final form as a sample textbook. Teacher reports on the grade 8 course were made chapter by chapter by some 100 teachers in the original ten Centers and a few additional school systems. No further work was done on the seventh grade course.

Some Immediate Reactions

Though it is too early to assess the long-term effects of the work of this group, a number of definite results are already apparent. Some parents object to the fact that their children are learning things which they did not have in school, but for the most part the parents are most enthusiastic. In fact, in some Centers, classes have been set up for parents who wish to learn about the materials themselves.

Books are beginning to appear--some superficially using some of the materials and others concerning themselves with the basic points of view of this and other curricular groups.

Formal Evaluation

In 1959-60 and 1960-61, evaluation of the use of the seventh and eighth grade materials has been carried out by the National Testing Center at the University of Minnesota. The evaluation in general has been quite favorable. One of the interesting first reports showed that students were divided into three ability levels: those of the highest and lowest ability made the greatest progress, those of middle ability making progress but not as significantly great as those in the first and third groups. At all times the materials have been intended to be course materials for all students in seventh and eighth grades, although in the Centers there have been a preponderance of students with I.Q.'s over 100 studying the materials. Some teachers have reported that certain chapters have been difficult for the slower students; but, again, the point of view has been that, if the slower students were given sufficient time, they could complete all of the chapters.

Relations with Other Writing Groups

The experience with the School Mathematics Study Group courses for grades 7 and 8 has provided a valuable background for writing materials for grades 4, 5, and 6; and it now appears that a considerable part of the course for grade 7 will appear in appropriately modified form in the text for grades 4, 5, and 6. This may mean that, in the future, the present School Mathematics Study Group course for grade 8 will become the regular seventh grade course, and that the School Mathematics Study Group course for grade 9, namely, the first year of algebra, might become the regular eighth grade course for at least the upper fifty per cent of students.

Major concerns of the writing teams for the School Mathematics Study Group materials for grades 7 and 8 have been the question of applications and the criticisms so often heard by teachers of other subjects that although pupils do "A" work in mathematics, they cannot solve the simplest problems in applications; for example, ratio and proportion in home economics, agriculture, or science. The writing teams being conscious of this problem and criticism have attempted through exercises to make this situation less critical. There also has been an attempt to introduce applications from the sciences, and in so doing to use the language and the point of view of the other curriculum studies in the sciences; for example, the chapter on the lever was rewritten in the summer of 1960 using the vocabulary of the physics course of the Physical Science Study Committee. A number of problems in the chapter on probability were actually written by members of the writing team of the Biological Sciences Curriculum Study working at the University of Colorado in Boulder.

Supplementary Units and Teacher Aids

Since there is a great wealth of topics which students in grades 7 and 8 can study, there have been from the beginning certain supplementary units. The texts themselves have been sufficiently long that too few teachers have had opportunity to use the supplementary units, although there is some evidence that unusually bright students have worked on these in an individual study situation. In a few instances a chapter regularly established in the book has been changed to a supplementary unit, and vice versa. The writing team members and the Panel are confident that the supplementary units are a valuable part of the total effort of the School Mathematics Study Group to improve the mathematics of grades 7 and 8. It is hoped that more of these can be introduced as teachers become more accustomed to studying the materials. Certainly, it will be possible to use more supplementary materials when students have studied some of the new materials of grades 5 and 6 before starting the work of grades 7 and 8.

Throughout the work there has been a concern about teacher commentaries and supplementary material for the teachers. In the SMSG commentaries appear not only the answers to problems but background information so that the teachers may have some idea about the reason for the introduction of certain topics--how they relate to what is to come after and from what point of view they are presented. An attempt is also made to point out and answer some of the questions which may be asked by the brighter pupils and to indicate what directions further study might take. Upon the recommendation of the Teacher Training Panel and the Panel for Grades 7 and 8, a text on geometry for junior high school teachers has been prepared and made available. In addition, a revised edition of the courses for grades 7 and 8 has been written for study by elementary school teachers.

HISTORICAL AND CRITICAL REMARKS
ON SMSG'S FIRST COURSE IN ALGEBRA

by

H. O. Pollak

1. Some Comments on the Working Technique of the Ninth Grade Team

a) A number of us began the summer of 1958 with the strong feeling that we should begin by taking a long fresh look at the problem of first-year algebra, and that we should not accept as either boundary or initial conditions the recommendations of any other group. Thus, we did not set out to follow any particular course outline which had been previously proposed, nor did we find it profitable to argue local variations of existing texts. The group had to find its own logic, and it took a fairly large chunk of new beginning material to get the project off and running: the outlining of a whole chapter on sets of numbers, sets under addition and multiplication, and closure. No attempt to follow a pre-existing outline, or improve a pre-existing text, could have infused the group with a comparable spirit of excitement and eagerness for the job ahead.

b) It was the philosophy of the Ninth Grade Group that we should not proceed with any portion of the text until everyone had really agreed on the content and the spirit of that particular portion. We thus spent a very large amount of time at the beginning, arguing rather than writing. This had the effect of making everyone happy with almost everything we turned out. I would not recommend a division of labor in which the subject matter is parcelled out among various groups, of say, two people each, after only a day or two's discussion. I think the inevitable consequence of this is that the rest of the Group will not agree with the production of any pair, that people will feel unhappy, and that the whole material will not hang together very well. Now of course you have to subdivide in some fashion in order to get the work done. I think the subdivision by function rather than by chapter is much to be preferred; a small group working on text, a group on teachers' commentary, a group on problems, etc., all on the same chapter at roughly the same time.

c) Although it was never established explicitly, it was in fact almost always true that the secondary school teachers had a veto power over the material which was written. It might be the loveliest imaginable mathematics, but if it was not, in the opinion of the teachers, teachable, out it went.

2. Other Possibilities for the Course

In the summer of 1960 a possible major change in organization of the beginning occurred to us, but it was too late to put this into effect, even if on further thought we would have wanted to. I do think, however, that it deserves to be considered by any other group going over the same ground. Our course begins by assuming knowledge of the arithmetic of non-negative numbers, and then proceeds to discover the properties of the operations, to introduce variables and to formalize the statement of the properties, all before the introduction of negative numbers. This is done in order to allow us to motivate the operations for the negative numbers from the operations as they are known for the positive numbers. I think it would be possible to begin the course by studying the arithmetic of the negative numbers and making this also part of the student's background. Then the properties of the operations could be pulled out for all the numbers at once. This approach might save some time and would also have the advantage of getting to negative numbers sooner than we now do. It would have the disadvantage of removing some of the "punch" from the notion of extension of an operation.

In the same line we were not at all happy about doing addition of the negative numbers before we did the multiplication. We had to use this order because we need the addition property of the opposite in the course of motivating the product of two negative numbers, and this is one of the keystones of the course. The reason that we did not like this order is that addition is so much more complicated than multiplication in its definition in terms of familiar operations. There are too many cases. Multiplication has a simple definition, and it would be much preferable it you could do it first. If arithmetic of negative numbers had been done at the beginning, this might perhaps have been possible.

3. Opinions of Specific Subjects

a) We had quite a big argument over the addition property of equality. It seemed to the mathematicians that this was not worthy of being pulled out as a distinctive property of the real numbers, since it was really just a semantic fact. If a and b are names for the same number, then $a + c$ and $b + c$ are of course also names for the same number. Thus the mathematicians hated to give it a place as important as, say, the corresponding property for the order relation. On the other hand, the teachers insisted that this occupied a major place in the solving of equations, and that they had to have something with a name and with some official stature that they could refer to. The final decision was in favor of pedagogy.

b) There was a lengthy discussion over the use of one order relation or two in the statement of, for example, the trichotomy. The usual statement of this is "given any two numbers a and b, exactly one is true: $a < b$ or $a = b$ or $a > b$". The mathematicians did not like this. Trichotomy should be a property of the order relation and not of the two numbers. Thus it would be mathematically preferable to say "given any two distinct numbers a and b, either $a < b$ or $b < a$" but not both. This way it is very clearly a property of the order relation "<". The teachers were not sure that they liked this because it isn't really the way in which the usual order relation is used.

c) We were rather proud of the technique used for introducing the student to word problems. Ordinarily these are one of the most-feared topics in an algebra course, but by all accounts we have a pretty good treatment of this subject. The problem is, after all, one of translating English into algebra. We get a lot of mileage out of translating from algebra into English first. "Give me an English interpretation of $2x + 3$." If thirty different students are each instructed to bring in three or four interpretations of such a phrase it is quite possible to cover all the usual interpretations of addition and multiplication in words. This is done first for phrases and then for sentences, and it is done simultaneously for inequalities as well as equations.

We are very happy about some of our word problems for inequalities and wish we had more. For example, "Two sides of a triangle are six and five inches respectively. How long is the third side?" is a very interesting problem. One of the things a student should begin to think about is whether he has enough data for an equation, or just for an inequality.

d) The order of the material in the first five chapters is dictated by some rather strong logical considerations which are worth remembering. The key is the introduction of multiplication for negative numbers. We wish to be able to cite the anticipation of the continued validity of the distributive property as the reason for the definition of this multiplication. This means that the distributive property for positive numbers, in a rather general formulation, must have preceded this point. Very well, this means that we must introduce variables before we introduce negative numbers, because the general formulation of the distributive property uses variables. But it turns out that we wish to use the distributive property in the first introduction of the variables! How do you get around this? We do it by first giving an intuitive feeling for the distributive property, with specific numerals and not requiring any formal statement. This then is used to play a number game, through which variables are introduced. The variables then are used to state the distributive property and also to enable us to speak of the truth set of an open sentence. Before doing an informal study of the distributive property, however, it is worth while to talk about sets of numbers and the number line. Thus the order of early material is settled. Sets of numbers and the number line must come first, informal feelings for the properties of addition and multiplication come second, then the introduction of variables, then a formal statement of the properties, then the introduction of negative numbers, and then the operations of them. We interrupt this sequence only to bring in the study of translation between English and algebra so that we can get this in early and can start to do some interesting problems.

e) There are really two ways in which the negative numbers can be considered. We may consider negative and positive numbers as signed real numbers and essentially as vectors. This makes them different beasts from the unsigned numbers of arithmetic. The alternative is to take the negative numbers simply as extensions of the familiar numbers of arithmetic and to make no fuss about signed vs. unsigned at all. For a number of reasons we chose the second of these methods. First, we don't think that the negative numbers as such are really new to the students. The operations on them may be, but the numbers themselves are part of their everyday experience. We have always drawn the number line with an extension to the left of zero and we consider the names for negative numbers in connection with names for points on the left half of the number line. Secondly, and this is more important, we felt that the introduction of two classes of numbers and the subsequent identifications of one class with half of the other class as unnecessary and befuddling. It would be very difficult to get across at this stage what the necessary isomorphism is all about. As the icing on the cake, you may consider the problem of defining an absolute value! This would have to be a mapping from the signed to the unsigned numbers; and yet in our work we wish to mix numbers and absolute values from the very beginning.

f) Speaking of absolute values, we had quite a discussion about the proper definition. We finally chose to define the absolute value of a number as the greater of a number and its opposite. The alternative under strong consideration was the positive of a number and its opposite. The argument in favor of the second of these is that it is closer to the way we use absolute values. The argument in favor of the first is that it does not get the idea of positive and negative into the definition. We wish to avoid this as part of a continuing program to keep students from thinking that $-x$ is always a negative number. Either of these definitions however is preferable to an analytic one from the very beginning. Absolute values are used throughout the course in more and more different ways. Almost every chapter after their introduction in Chapter 5 finds use for them. In Chapter 6 absolute values are used in connection with the definition of addition. In Chapter 7 they are needed to define multiplication. In Chapter 9 we speak of the distance between two points in terms of absolute values. In Chapter 11 they are needed in defining the square root properly. In Chapter 12 they are used in thinking about factoring quadratics. And in Chapter 13 they form the simplest examples of equations with extraneous roots. Finally, graphs of open sentences involving absolute values and the absolute value function occur in the last three chapters. Thus, the concept of an absolute value is enriched quite considerably as the course progresses.

g) One of our principles has been that we should be careful about how we say things. It was our feeling that the poor quality of standard texts is in part due to their imprecise definitions and sloppy statements. Thus, for example, we maintain the distinction between numbers and numerals for some time and in fact find it very useful in connection with consideration of simplification. It is possible to maintain the distinction in connection with fractions because there are still two English words. We may use "fraction" for the numeral and "rational number" for the number. This breaks down, however, when the time comes to speak about numerators and denominators. Are these numerals or are they numbers? There is just one English word. At this point we simply drop the fine distinction and admit that the student will be able to tell the difference. There is no point in tangling yourself up in this language for very long after the student has seen what you are driving at. The important idea is that you have to earn the right to be sloppy. This means that you must be able to stop in any derivation at any point and know why it is valid. But every students must certainly become proficient enough for mechanical manipulation in which the distinctions are forgotten.

h) The mathematical meaning of the various problems captioned in the textbooks by "simplify" is interesting. This is a typical vague instruction which the good students in the past have learned to follow correctly by instinct and which the poorer students never have understood. What is meant is that for certain kinds of expressions there are unique preferred forms. The existence and uniqueness of this standard form is a mathematical theorem; the student must then learn to find it. For example, it is a theorem that no expression need contain more than one indicated division, and the reduction of an expression to just one indicated division is the content of such topics as addition of fractions, multiplication and division, and compound fractions. Similarly, there are standard forms for expressions involving radicals. Once you have the theorem that the square root of any integer which is not a perfect square is irrational, you know that an expression like $1 + \sqrt{2}$ cannot be simplified further. It is another theorem that $\sqrt{3} - \sqrt{3}$ cannot be simplified further. Again, there are standard forms here, and the instruction to simplify means exactly to reduce to one of these forms. In order to get all this across, the notion of different names for the same number is very handy.

i) We had quite a lot of discussion about solving of sentences, particularly equations, both with regard to the mathematical meaning of this, and its place in the course. Every sentence has a truth set. Some very simple sentences have obvious truth sets. There are operations which can be performed on sentences which leave the truth set invariant. The point of solving an equation, for example, is to change it, by operations which leave its truth set invariant, into an equation whose truth set is obvious. Since this is the mathematical content, it should come after all the operations which leave truth sets invariant, and this means after the chapters on addition, multiplication and order of the real numbers. At this point, however, the teachers, with right, objected. This is much too late in the course to be solving equations. Consequently, the introduction of equation solving techniques begins much earlier, and the material is finally brought together in a chapter on truth sets of open sentences. There are of course equations which cannot be changed into one with an obvious truth set by operations which do not alter the truth set. In this case you are willing to perform operations which increase the truth set. No solution is then lost, but is is necessary to check at the end that any values found are actually solutions. It is also necessary to check solutions of quite simple equations before the invariance of truth sets under addition and multiplication has been established.

j) It is very worthwhile to have a chapter of factoring of numbers before the factoring of polynomials. There are quite a number of reasons for this. In a kind of philosophical sense the additive structure of numbers is really sufficient for most of arithmetic, while the multiplicative structure is essential in algebra. Thus it is sufficient for arithmetic to know that $288 = 2(100) + 8(10) + 8(1)$, while there are many places in algebra which require $288 = 2^5 \cdot 3^2$. In particular, we may cite work on simplification of radicals, and work on lowest common denominators. It is also useful to have work with factoring of numbers to motivate the work on exponents. Most important, however, is the fact that it is absurd to ask a student to factor $x^2 + 11x + 24$ if he cannot factor 24. We are also rather happy about the insight into quadratic factoring which is obtained through consideration of the prime factorization of the coefficients. This is a much better method than the usual outright trial and error.

k) Our feeling about introducing new symbols and new terminology was that we do not want to have them until the student really wants and needs them. We introduce only one non-standard notation as a straw man, and that is a symbol for the opposite. This is essential to distinguish initially the three uses of the minus sign. We have not found it essential to do away with the common terminology of substitution and even variable. You simply cannot hide from a student the way everyone else speaks. What you can do and have to do is to be very careful to define and use things precisely. In a similar vein, we do not insist on notation for the union and intersection of sets and on the set builder notation. These ideas do not occur sufficiently often to make the notation necessary. If the teacher would like to introduce it, of course this is all right.

l) The first theorem in the course occurs in Chapter 6 and is the uniqueness of the additive inverse. We felt the need for this theorem very strongly, but we also found it a difficult theorem to begin with. It is really hard to get across to students what you are fussing about. At one stage in the summer of 1959 we even wrote a little play (called Chip and Dale) because what we really needed was somebody to ask the right stupid question at a couple of places. We didn't have the courage to keep this in the final version, and wrote a lot of words to try to cover the same point. It would be nice if something a little easier conceptually could be the first proof. As a matter of fact, we wrote several proof-like arguments earlier into the text (see for example page 32) so that the student would be prepared a little better. There are quite a number of theorems and proofs in the course. We, of course, do not expect complete appreciation of the nature of theorem and proof from the student from the beginning, or even by the time he gets to the end of the course. We do want the student to become acquainted with the idea of proofs in algebra and to meet some of the different kinds of theorems, uniqueness, existence, if and only if, reductio ad absurdum, and so forth.

m) One of the things which is forever going on in mathematics is the extension of a set of elements in order to make it closed under an operation, and then the subsequent extension of the definition of other operations in order to give their meaning on these new elements. Thus we define negative numbers in order to complete numbers under subtraction. We are then faced with the problem of defining addition and multiplication for this larger set of numbers. A similar thing begins to happen with exponentiation. This is defined initially for positive integer exponents. We first extend to negative exponents, and later even begin to meet fractional exponents. Actually the logical completion of this sequence, complex numbers to complex exponents doesn't come until very much later. The point is always that you wish to use operations previously defined in order to extend these operations, and you require that whatever properties you had before to continue to hold. Thus, for example, multiplication for negative numbers involves the three notions of opposite, absolute value, and multiplication for the nonnegative numbers, all of which were previously defined.

n) In the chapter on functions at the end of the course, we do not go so far as to define a function as a set of ordered pairs. What we do is to talk about many different ways of defining a correspondence, such as a verbal description, a table, a graph, a formula, a Begle meat grinder, and we point out that what is important is the correspondence itself and not the way in which it is defined. The correspondence is defined to be a function. We felt that this was the basic idea which we wanted ninth graders to have. It would, of course, have been possible to go on to the definition in terms of sets of ordered pairs, but functions are, after all, taken up again in all succeeding grades. We felt in this case it was more important to build up a good intuition than to try to tell the whole truth from the very beginning.

o) The ninth grade course contains a lot of work with quadratic functions and much material which shows you how to move a parabola this way and that, to open it up, etc. We do not present the quadratic formula as such. There is good reason for this. We do not want the students memorizing a formula they do not understand, and we want them to have a good geometric picture of the significance of completing the square. In addition, it is even in practical work frequently better not to use the quadratic formula because as you complete the square you have a chance to do simplification on the discriminant, while the rest of the mess is still off on the other side of the equation.

p) The exercises play a very key role in this course. We wanted them to be used for discovery a great deal and we wanted to avoid long strings of purely repetitive problems. Thus, in a good set of exercises, each question might well contain some little thing a bit different from all the others. In successive exercises more and more items should be brought out until the last exercises ask the students in fact to begin to derive in his own mind what comes next in the course. We originally had a tendency to go too far in this direction, and not to practice a relatively difficult new idea sufficiently. Unfortunately, it is remarkable how difficult it is to write a good set of exercises. I am sure this was a revelation to at least some of our writers. It takes an enormous amount of time and care to put together a set of exercises to do the thing we are describing. It is much more difficult than writing text.

q) One thing which you have to work very hard to overcome is the tendency of some teachers to latch on to iron-clad rules for the students to follow absolutely. These teachers, for example, insist on the way to do a problem, while we wish the students to think freely and attack a problem by any technique whose relevance comes to mind. We want the students to think about their mathematics and enjoy it, not to memorize a bunch of rules and glorify the ans instead of the understanding. Thus there are places in the text where we specifically ask the student for more than one way to do a problem and to think about the relative merits of the several methods (e.g. solving of simultaneous equations). There are other places where we warn the teacher specifically against one of the old fashioned principles which is just not good mathematics (e.g. always rationalizing a denominator).

x) Last, but not least by any means, we come to the problem of variables. We certainly had an immense amount of discussion on this point, and changed our minds several times. Initially we believed in the place-holder concept; in order to make the discussion precise, let me say exactly what the difference between a variable and a place-holder in our minds really was. A place-holder is a symbol which holds the place for the name of a number, which number is to be selected from a specified set. A variable is a symbol which is the name of a number, which number is to be selected from a specified set. Some of the chief arguments on this point are

1. It is well known and accepted that the notion of a place-holder is pedagogically very sound. It is a good notion for teaching purposes.

2. The place-holder idea is used by logicians in their current codification in the foundations of mathematics.

3. Mathematicians, engineers and almost everybody else actually use variables in much of their work. We can see this in the way we think of "$x + 3$" along with "$2 + 3$". We can see this in our attack on word problems. We can see this in our definition of certain algebraic numbers such as the real solution of $x^5 + x + 1 = 0$.

4. Questions with regard to truth sets of open sentences, for example, are quite the same with both definitions.

5. There are other uses of symbols such as "x" in mathematics. We already meet an indeterminate in this course when we define polynomials. Later in his mathematics career the student will meet the identity function, and may also learn to think of a variable as a function whose range is a given set. Thus, there is no notion which covers all cases.

6. There are some excellent devices for teaching the notion of a variable as we wish to use it. We particularly emphasize games with numbers. A student begins with a particular number, performs a sequence of operations and always seems to come back to the number with which he started. It is very natural to carry out the instructions without specifying the chosen number in advance. Thus the idea of a variable is on the way.

With these points and many more in mind, we decided finally to stick with variables rather than place-holders. The name "variable" does pain some people but we are sure that we can teach it clearly. Certainly, this is a point on which the argument is by no means settled.

THE SMSG GEOMETRY PROGRAM

by

E. E. Moise

1. Content

The SMSG geometry text is designed to be used in a year course, usually given in the tenth grade. It includes enough plane synthetic geometry for about six months' work, with shorter introductions to solid geometry and analytic geometry. The choice of material, within these time limits, is approximately what the reader might expect. (For a list of chapter titles, with a brief account of the contents of each, see the end of the present article.)

It will be recalled that when the SMSG began its work, at Yale, in the summer of 1958, the total writing group was split into four teams, one for each of the high school grades. As a provisional basis for this division of labor, the writing teams agreed that they would cover approximately the material recommended in the report of the Commission on Mathematics of the College Entrance Examination Board. In the case of the tenth grade, the approximation was extremely close. With minor exceptions, all of the Commission's topics are covered in the SMSG book. (The only exceptions that seem worthy of mention are spherical geometry and solution sets of linear inequalities.) In fact the only major difference between the two programs, as far as content is concerned, is that the SMSG book gives proofs for many theorems for which the Commission suggested merely informal indications of proof.

In its style and spirit, however, the SMSG book is in many ways quite different from other programs both current and proposed; and some of these differences are not easy to explain briefly. Let us begin by comparing various alternative treatments.

2. The Euclidean Program

In the context of modern mathematics, the treatment of geometry given in the Elements is in many ways rather peculiar. Here we are not referring, in a negative sense, to the logical defects in the treatment, for example, the refusal to admit that some terms are undefined, the omission of postulates governing betweenness and plane separation, and so on. (We shall discuss these matters later.) We refer, rather, to the form that Euclid's theory takes when these defects have been removed, as, say, in Hilbert's *Foundations of Geometry*.

In Euclid, the algebra of the real numbers is totally absent. In fact, the only numbers mentioned in the *Elements* are the positive integers. This means that there is no such thing as the length of a segment. Instead there are relations of "equality" and "greater than," between pairs of segments. Thus, in effect, we have the concept of "same-length" and "greater-length," but not the concept of length. The length of a segment would have to be a real number, and real numbers do not appear.

This leads to certain delicacies and complications, even in the treatment of congruence and inequalities. And in the treatment of similarity, the situation becomes more difficult still. Given a triangle whose sides are the segments A, B, C, and a similar triangle whose corresponding sides are A', B', C', we would like to say that corresponding sides are proportional. That is,

$$A : A' :: B : B' :: C : C' .$$

If we were talking about *positive numbers*, the meaning of this proportionality would be clear enough; it would mean simply that

$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'} .$$

But it is not easy to see what the quotient of two segments ought to be.

Euclid's way around this difficulty was ingenious and subtle in the extreme. It was as follows.

Let A and A' be segments, and let p and q be positive integers. Suppose that if p "equal" copies of A are laid end to end, and q "equal" copies of A' are laid end to end, the first of the resulting segments is shorter than the second. Then we write $pA < qA'$.

We can now explain what is meant by the proportionality

$$A : A' :: B : B' .$$

It means that $pA < qA'$ if and only if $pB < qB'$, for every p,q.

This was Euclid's working definition of proportionality. The interested reader may be able to convince himself that the definition works. A thorough mathematical analysis indicates that the real numbers are implicit in this scheme, and that in effect, Euclid was describing real numbers by means of Dedekind cuts in the rational numbers. But the mathematical theory involved here is quite difficult; and it seems plain that Euclid's non-numerical methods are unsuitable for the purposes of tenth grade instruction.

3. The Metric Treatment

In this alternative treatment, the real number system appears at the outset. It is assumed that the distance between pairs of points is given, so that segments can be measured; the length of a segment is the distance between its end-points. (There are, of course, analogous postulates for degree-measure of angles; in discussing segments, we are merely trying to illustrate the spirit of the metric scheme.) Thus "equality" for segments is not undefined; it means simply that the segments have the same length. In a metric treatment, it makes sense to label the sides of a triangle with numbers, such as 1, 2, $\sqrt{2}$, and so on; these labels indicate the lengths of the segments; and segments really do have lengths, in the mathematical theory that the figures are supposed to illustrate. The problem of defining proportionality, for similar triangles, becomes trivial. If the lengths of the sides of the first triangle are the numbers a,b,c, and the lengths of the corresponding sides of the second triangle are a',b',c', then the proportionality relation means simply that

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} ;$$

and from here on, the treatment of proportionality is entirely algebraic.

One of the postulates, in the metric treatment, says that on any line we can set up a coordinate system, in the usual sense. This makes it very easy, at the appropriate point, to set up a Cartesian coordinate system in the plane. (To do this on the basis of the purely synthetic postulates of Euclid or Hilbert requires that we prove an extremely difficult theorem, whose proof is far beyond the grasp of a high school student.)

4. The Pseudo-Synthetic Presentation

As the reader is no doubt aware, there is by now nothing novel about the metric treatment. Universally, elementary geometry books talk about lengths of segments and degree-measures of angles; they give figures in which both segments and angles are labelled with the numbers that measure them. The Euclidean concept of proportionality is by now hardly more than a historical curiosity, not very widely remembered or understood even by professional mathematicians. Universally, elementary texts discuss proportionality in the algebraic terms that we have just been discussing. All this, we believe, is as it should be; it vastly simplifies the theory, and it brings the theory closer to the methods and spirit of modern mathematics. The strange aspect of the matter is the fact that the currently "conventional" books, for reasons which appear to be purely historical, undertake to do metric geometry on the basis of synthetic postulates. The resulting treatments may fairly be described as pseudo-Euclidean. In such treatments, the use of Euclid's language, and the use of his purely synthetic postulates, are hardly more than gestures. And they are rather misleading gestures, because they do not describe the mathematical system which is in fact going to be studied or the methods which in fact are going to be used.

Pseudo-Euclidean treatments, in elementary study, necessarily involve a logical gap: since distance is not given in the postulates, we need to show that it can be defined; and the proof is beyond the grasp of nearly all young students. (Any valid proof must appeal to the Archimedean property of lines and the completeness of the real numbers in the sense of Dedekind, because without these hypotheses, the theorem involved becomes false.) The reader will observe, however, that except at this one point, the above discussion has no connection, one way or another, with the question of logical rigor. Any treatment of geometry (Euclidean, metric or pseudo-Euclidean) can be taught at any desired level of rigor, however high or low.

We now proceed to consider how the concept of rigor ought to apply in elementary instruction.

5. Accuracy and Clarity

If we consider the offenses against rigor which are now current, it is plain that the neglect of conceptual subtleties is the least of our worries. The simplest illustrations of our worst worries are to be found in algebra. In an algebra text now in print one finds the following sentence:

"So we see that the square root of 64 is ± 8."

It would seem easy enough to say:

"So we see that the square roots of 64 are 8 and -8."

But the shorthand "±" has led the author to claim in one breath--while denying in the next--that 64 has only one square root. An examination of the text literature indicates that this is not an isolated or a casual lapse. For example, not uncommonly we find in bold-face type the displayed formula

$$x^2 = x .$$

This holds if and only if $x \geq 0$. That is, the displayed formula is about as valid as the statement that on a checkerboard, all the squares are red. It would seem easy enough to write

$$x^2 = |x| .$$

But this seems to be a major reform; in most books, if you want to get the answers given in the answer section, you must "simplify" x^2 and get x ; and above all, you must "simplify" $x^2 - 2ax + a^2$ and get $x - a$.

The same sort of issue arises in geometry. Not uncommonly, an angle is defined as "the figure formed by two intersecting lines." (Obviously, two intersecting lines form at least four angles; and if you cound "straight angles" and "zero-angles" there are six more.) Sometimes an angle is described as an "amount of turning," thereby inviting confusion between a geometric object and a number which in some sense measures it. The distinction between lines and segments is simple and intuitive: segments are bounded figures, with end-points, while lines extend infinitely far in two directions. To speak of "extending a segment" makes sense; to speak of "extending a line" does not. Nevertheless, lines not uncommonly get extended in textbooks. Even in a "modern" treatment we find the statement: "A straight line and a plane are parallel if they do not meet, however far they may be extended."

Considering the prevalence of this sort of writing, it is easy to understand the difficulties that students have when they are finally expected, at some point, to read and understand exact mathematical statements. The trouble is not merely that they have failed to acquire a positive skill. They have, in addition, acquired a negative skill and a deep skepticism: they have learned, by long experience, that statements made in textbooks cannot necessarily be taken literally, and cannot be relied upon to reward careful examination.

6. Deductive Rigor

It may be true, in some philosophical sense, that there is such a thing as absolute deductive rigor. But absolute rigor is not a part of the working equipment of most professional mathematicians. In fact, unless a mathematician happens also to be a logician, he is likely to do set-theory from the "naive" viewpoint, without the use of postulates. (A distinguished mathematician has recently published a graduate-level textbook under the bold title Naive Set-theory.) If we view mathematics realistically, as a human activity, it is plain that there are degrees of mathematical understanding; imperfect understanding is the kind that occurs most commonly, and is the kind that occurs first, in the development of an individual. For this reason, no teacher need feel ashamed of bringing students merely to the level of understanding that they are capable of achieving at a given stage of their education. In a discussion of this question, at a meeting of the SMSG geometry group in 1958, a formal motion was made to the effect that logical rigor is not a moral matter. The motion passed without dissent.

On the other hand, deductive reasoning is an essential method in mathematics; and it ought to be done as carefully as pedagogic practicality permits. One way of putting it is that deduction is the laboratory of mathematics. The analogy goes rather far. It is clear that if you "teach" chemistry without using a laboratory, then you can "cover" quite a lot of material in the time that you "save." But a long accumulation of hearsay, from a higher authority, is borrowed knowledge: the student does not own it and does not keep it. For this sort of reason, mathematics should be taught, whenever possible, by sound and convincing proofs. The student should feel that he understands, in his own mind, why the theorems that he learns are true.

It should be borne in mind, on the other hand, that logical exactitude is not the only essential ingredient in mathematical understanding. Just as chemical experiments are designed by human minds, rather than by laboratory equipment, so mathematical ideas originate by creative intuition: without such intuition, we have nothing to verify with our logic. These remarks apply, in the same way and for the same reasons, to the research done by a mature mathematician and to the problem work done by a tenth grader.

These general views are rather close to those of the Commission. In their report (p. 22 and following) they remark that the teaching of geometry has three main objectives. "The first objective is the acquisition of information about geometric figures. . . . The second . . . is the development of an understanding of the deductive method. . . . Deductive methods are taught primarily to enable the pupil to learn mathematics. . . . A third important objective of the geometry course is the provision of opportunities for original and creative thinking by students."

So far, the SMSG geometry group agreed. But it would be an exaggeration to say that our view coincided entirely with those of the Commission. A little later, the Commission said that "There are essentially three defects in the Euclidean development of geometry that make it unsuitable as the basis for modern high school instruction. . . . The first defect is Euclid's failure to formulate explicitly the axiomatic basis on which congruence theorems rest. . . . The second defect is . . . the lack of an adequate algebra. . . . The third . . . is the failure of Euclid to recognize the necessity of making formal assumptions concerning 'betweenness.' . . ."

Here it appears that the question of rigor is being overemphasized. Euclid was very rigorous indeed, by high school standards. The main difficulty, rather, is that the "lack of an adequate algebra" goes so deep: the whole conceptual apparatus of Euclid is incongruous, in the context of modern mathematics.

The Commission went on to say the "one possibility is that of developing a logically unimpeachable treatment of Euclidean geometry suitable for use in secondary schools. Although some distinguished mathematicians . . . have undertaken this task, the Commission feels that at present no presentation suitable for high school use exists. . . . The Commission hopes that mathematicians will continue to strive for the development of a logically sound basis for Euclidean geometry in a form suitable for high school use. However, for the present, it recommends that textbook writers and teachers should feel free to modify the Euclidean development to attain a more incisive and interesting program."

Here the ultimate task assigned to mathematicians appears to be unreasonable: we never expect to see a logically unimpeachable elementary book. Our own book is very impeachable indeed. The treatment of betweenness is short and sketchy. Logically necessary separation theorems are not proved, or even stated. It is assumed at the outset that the real numbers are "known." Everybody is aware that they are not known very thoroughly, to students entering the tenth grade; and they are not adequately explained in our book. It appears to us that any pedagogically sound book is very likely to be open to this sort of rigoristic approach.

The Commission seems to have regarded this whole question with mixed feelings. At one point they remark that "it is now possible and desirable to use the deductive method in all mathematical subjects, and consequently the time devoted to it in geometry can be somewhat reduced." Here our disagreement runs in precisely the opposite direction: if it is true that "deductive methods are taught primarily to enable the student to learn mathematics," then deductive algebra will merely make deductive geometry easier, without making it less desirable.

The sample text material in the Commission's appendices followed the lines suggested by the last quotation, rather than the earlier ones. Thus the relation between theory and practice, in the Commission's work, as compared with ours, is far from simple.

7. The Role of Analytic Geometry

The treatment of analytic geometry in the SMSG book is somewhat more careful than most, in keeping with the method and spirit of the rest of the book, but it includes no novelties deserving special mention. In this section we merely explain how, in our view, the chapter on analytic geometry is related to the rest of the book.

Some have expressed the view that Cartesian coordinate systems enable us to solve, or to avoid, the problem of the foundations of geometry. In a sense, this is true. It is possible to set up geometry by (1) defining a point to be an ordered pair of real number, (2) defining a line to be a set which is the graph of a linear equation, (3) defining distance by the usual distance formula, and then building on this structure, without using any postulates at all. This approach is usually avoided, and with good reason. In the first place, it is slow and tedious. In the second place, it yields a rather poor exposition of the facts. In elementary geometry, a few simple statements are basic and crucial; these are the statements commonly taken as postulates; and one of the expository virtues of the postulational treatment is that it brings these key statements plainly into the foreground.

Almost universally, of course, Cartesian coordinate systems are set up on the basis of synthetic postulates. Under this scheme, unfortunately, they have nothing to contribute to the problem of the foundations. The easiest way to see this is to recall that to set up coordinate systems at all, and prove even the simplest theorems about them, we need to know the rudiments of the theory of congruence, parallelism, perpendicularity, and similarity, plus the Pythagorean Theorem. By the time we get this far in synthetic geometry, the whole problem of the foundations is over with, for better or worse, usually worse.

It remains to consider the sort of contribution that coordinate systems make to the substance of the geometry.

The significance of Cartesian coordinates in elementary study is rather different from their significance to Descartes. Descartes invented them in order to bring algebra to the aid of geometry at times of dire need; he stated very difficult theorems, in purely synthetic terms, set up a coordinate system, and used it to get a proof. This sort of procedure does not figure very prominently in elementary study. In the usual college courses, nearly all of our attention is devoted to the study of the relations between the underlying geometry and the analytic apparatus. We show that certain figures are the graphs of certain types of equations, and vice versa. At many points, of course, we get really new geometric information. For example, if an ellipse is defined by means of a focus and a directrix, then it turns out that the ellipse has another focus and another directrix; and we can show that the sum of the focal distance is constant.

There are occasions, even at the beginning, where analytic methods can be used to advantage in proving geometric theorems; they enable us--for better or worse--to replace ingenious ideas by straightforward calculations. For example, the concurrence theorem for the medians of a triangle can be reduced to a calculation, and it is much easier to carry out the calculation than to devise a synthetic proof.

Such occasions, however, do not come up very often in the elementary study of geometry. Most of the time, when we "prove synthetic theorems by the analytic method," the main benefit is the practice that we get in using coordinates, and in relating them to the underlying geometry. With a few exceptions, as noted, the elementary theorems that lend themselves best to analytic treatment tend to be difficult exercises, rather than basic theorems that the student really need to know. (For example, the student has no urgent need to know that the segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.) Moreover, if we try too hard, and too soon, to use coordinate systems, we may drift into applying them in cases where they are absurdly inappropriate. Consider, for example, the following theorem:

Two lines perpendicular to the same line are parallel.

Synthetically, this is trivial. If L_1 and L_2 are perpendicular to L, and L_1 intersects L_2 at P, then there are two perpendiculars to L, through P, which is impossible. Note that if you don't know about the uniqueness of perpendiculars, then you can't set up a coordinate system in the first place, because you can't explain what is meant by the x-coordinate of a point.

To prove this theorem analytically, we would use the formula $m' = -1/m$, for the slopes of perpendicular lines. Thus, if we are not very careful, we will get not only a circuitous but an incomplete proof, covering the case where none of our lines are vertical.

For these reasons, the SMSG group considered that analytic geometry ought to be regarded, for practical purposes, as a new subject and not as an integral part of the basic introduction to geometry; and we therefore covered it in the last chapter. (The last chapter need not be studied last. As we pointed out in the commentary, it can be taken at any point after the chapter on similarity.)

It should be understood that our order of presentation was not based on a low opinion of the importance of analytic geometry. Its importance is both immense and unmistakable. Therefore it must be taught. But from this it does not follow that it must be taught in a given month of a given course.

There are possible disadvantages in our choice. In practice, the course may profit from a change of style and tone, somewhere in the middle. If analytic geometry is introduced early, it is easier to give the student lots of practice in algebra. Finally, it may be that there is a category of students who are incapable of mathematical reasoning, but are capable of solving cook-book problems. ("Find the equation of the line which passes through the point (1,5) and is perpendicular to the line through (3,4) and (7,9).") For such students, a suitable treatment of analytic geometry may be a great relief. But we hope that this category is small. And on balance, we believe that our choice in the order of topics was a wise one.

8. The Use of Language

Probably the most striking novelty in our book, to most readers, is the terminology. We were aware that changes in terminology would cause trouble for teachers; and we therefore tried to hold them to a minimum, making changes only when the need for them seemed compelling. Nevertheless, our minimum turned out to be large; and should therefore explain our reasons.

(1) Congruence and Equality. Intuitively, two figures are geometrically equivalent if one can be moved onto the other by a rigid motion. That is, two figures are geometrically equivalent if they differ (at most) only in position, and have the same size and shape. For example, two segments are geometrically equivalent if they have the same length; two triangles are geometrically equivalent if they are congruent, and so on. (The technical definitions differ, for various figures.)

Frequently, of course, we want to make the stronger statement that two figures coincide entirely. For example, if P is the bisector of the base of an isosceles triangle, and Q is the foot of the perpendicular from the apex, then $P = Q$; that is, P and Q are merely different names for exactly the same point. This relation is the logical identity.

The logical identity occurs also in algebra. When we write

$$7 + 5 = 3 \times 4 ,$$

this means that $7 + 5$ and 3×4 are descriptions of the same number (namely, 12).

One of the troubles with the traditional language of geometry is that these two basic ideas are related in a needlessly complicated way to the words used to describe them, in connection with various objects. We may describe the prevailing usage by the following table:

Relations:	Objects:	Word:
The logical identity	Numbers	Equal
	Points	Coincide
	Lines	Coincide
Geometric equivalence	Triangles	Congruent
	Segments	Equal
	Angles	Equal

In this language, there are two words for the logical identity; there are two words for geometric equivalence; and one word is used in two quite different senses.

In our book, the correspondence between these words and ideas is one-to-one. That is, the logical identity is always described by the word *equal*; and the word *equal* is never used in any other sense. Geometric equivalence is always described by the word *congruent*; and the word *congruent* is never used in any other sense. The resulting simplification is considerable. It would be needless to review the way in which the old language developed. We believe that in the context of modern mathematics the old language has habit--and nothing but habit--to recommend it.

(2) *Congruences Between Triangles*. The expression

$$ABC \cong DEF$$

is sometimes interpreted merely to mean that the two triangles are congruent. It rarely happens, however, that this is the idea that we really have in mind. Nearly always, we go on to infer something about "corresponding sides" or "corresponding angles." Thus, what we really meant was not merely that the triangles are congruent, but that they are congruent in a particular way, that is, under a particular correspondence. In our book, this tacit understanding is made explicit. When we write

$$ABC \cong DEF ,$$

this means that the correspondence

A   D

B   E

C   F

preserves lengths of sides and measures of angles. Under these conditions, we describe the correspondence by the shorthand

ABC   DEF ,

and we say that the correspondence is a *congruence*.

It should be emphasized that the idea of congruences as correspondences is not an innovation. Our innovation consists merely in the fact that we speak openly about the idea that everybody had in mind all along.

(3) *The Language of Sets*. It is by now rather fashionable to introduce the language of sets into high school courses. We believe that once it has been introduced, it ought to be used in the cases where it rightly applies. In a plane, the circle with center P and radius r should be defined as the *set* of all points Q whose distance from P is r .

The *graph* of an equation, in a coordinate plane, should be defined as the *set* of all points whose coordinates satisfy the equation.

Similarly, the familiar "locus theorem" for perpendicular bisectors should be stated in the form: "The perpendicular bisector of a segment in a plane is the *set* of all points of the plane that are equidistant from the end-points of the segment."

It should be understood that here we are not using the "theory of sets"; we are merely using the language of sets. In the context of modern mathematics, the word *locus* is a superfluous synonym, serving only to complicate the language and obscure the unity of the subject.

Traditionalists have accused modernists of pursuing sets as a pointless fetish. If we introduce the word *set*, and then omit to use it in talking about the substance of mathematics, this charge becomes true.

9. Practicality

The above is a description of objectives. It was not predictable that these objectives would be achieved. In many ways, our book was ambitious. We were trying, harder than most of our predecessors, to convey to the student what this particular body of mathematics looks like to a modern mathematician. In addition to this, our innovations in matters of language could be expected to cause trouble for teachers, by asking them to change habits of long standing. We therefore had reason to fear that our book might be impractical in the classroom, or unacceptable to teachers, or both.

The reception of the book has been reassuring. It has gotten approximately its share of the total circulation of the SMSG's sample texts. The preliminary editions of these were used in some fifty-odd Experimental Centers, in 1959-60. The revised editions were placed on sale, starting in 1960-61. In 1960, about 100,000 copies of SMSG books were sold; and in 1961 this total increased to about 375,000. These figures suggest that the SMSG as a whole, and the geometry group in particular, have worked within the limits of practicality.

10. An Outline by Chapters

We now list the chapters in the SMSG geometry book, with some brief indications of their content, indicating the length of each, to give a rough idea of the emphasis given to various topics.

Chapter 1. Common sense and organized knowledge. A brief introduction to the postulational method. 14 pages.

Chapter 2. Sets, real numbers and lines. An informal discussion of sets and real numbers, followed by an introduction to the geometry of a line, based on the ruler postulate. 37 pages.

Chapter 3. Lines, planes and separation. 17 pages.

Chapter 4. Angles and triangles. 25 pages.

Chapter 5. Congruences. Note the plural. 58 pages.

Chapter 6.	*A closer look at proof*. A discussion of various questions which pertain to the preceding chapters, but which seem unsuitable for the beginning of the course. 30 pages.
Chapter 7.	*Geometric inequalities*. 30 pages.
Chapter 8.	*Perpendicular lines and planes in space*. 21 pages.
Chapter 9.	*Parallel lines in a plane*. 50 pages.
Chapter 10.	*Parallels in space*. 25 pages.
Chapter 11.	*Areas of polygonal regions*. 41 pages.
Chapter 12.	*Similarity*. This makes essential use of area-theory, at the start; hence the order of chapters. 45 pages.
Chapter 13.	*Circles and spheres*. 52 pages.
Chapter 14.	*Characterization of sets. Constructions*. A study of "locus" theorems, and constructions with straight-edge and compass. 44 pages.
Chapter 15.	*Areas of circles and sections*. 27 pages.
Chapter 16.	*Volumes of solids*. 33 pages.
Chapter 17.	*Plane coordinate geometry*. 64 pages.

THE PHILOSOPHY OF THE ELEVENTH GRADE WRITING GROUP

by

Frank Allen

When a mathematician looks at a school mathematics text, he is likely to be appalled by the superficial and inaccurate expositions it contains. He observes that even those texts which feature the so-called "meaningful approach" often used rationalization rather than proof, and fail to make a distinction between facts to be assumed and facts to be proved thereby obscuring completely the structural character of the subject. Thus, to the mathematician, most school mathematics texts appear as dreary collections of drill exercises ideally designed to repel the able students and useful only for the purpose of imparting specific manipulative skills by the intensive application of a large assortment of unrelated rules. Moreover, the authors of these texts seem to have no awareness of the patterns of reasoning which characterize mathematical thought or of the implications which the explosive growth and manifold applications of twentieth century mathematics have for the secondary curriculum.

By 1956 concern about the state of secondary mathematics had become fairly widespread among mathematicians--or at least among those who had bothered to look at the facts.

When the eleventh grade writing group, consisting of mathematicians and teachers in about equal numbers, met to consider its task in June, 1958, it was acutely aware of the fact that a good many influential mathematicians had bothered to look; that their concern had launched a well-financed drive to improve school mathematics and that some mathematicians, including some members of the eleventh grade group, were actively participating in this drive.

The initial discussions revealed that the group as a whole subscribed to the views expressed above. Some of the teachers however emphasized the importance of manipulative skills; noted that postulational development of mathematics at the eleventh grade level might seem a bit austere to many pupils and to some teachers; suggested that the idea of mathematics as a logical structure, conceived by the mathematician and worthy of study for its own sake, would run counter to the view, strongly held by many, that mathematics is fundamentally a tool subject which is best understood when studied in the context of its application. Others warned that nothing is more futile than presenting logical niceties to a class that does not have the background or the maturity to appreciate them. After discussion it was generally agreed that these viewpoints merited careful consideration. The wisdom of having a balanced team composed of mathematicians and teachers began to be apparent and the committee settled down to its task confident that, within its membership, there was a spokesman for any viewpoint which might deserve consideration.

Philosophy

The philosophy of the eleventh grade writing group developed as the writing progressed. The group hopes that the following viewpoints are discernible in the finished text.

1. The purpose of this text is to give the student some insight into the nature of mathematical thought as well as to prepare him to perform certain manipulations with facility.
2. If the student is to grow in mathematical understanding, he must encounter some ideas which are difficult for him. Such inherent difficulties must be candidly appraised and forthrightly explained in terms appropriate for students at this grade level. For this purpose the easiest and shortest explanation is not always the best. For example, the rules for solving systems of equations could have been given in much less space than it required in Chapter 7 for the development of the idea of equivalent systems; but this development provides a logical basis for understanding such rules.
3. Plausible arguments have their place provided they do not implant ideas which must later be eradicated. The necessity for improving the students' understanding of the nature of mathematical proof does not imply that everything must be proved or that proofs should be unduly rigorous.

4. It is often desirable to appeal to the students' intuition and to lead him by an inductive approach to make and test conjectures about the nature of the principles to be proved.
5. New symbolism should never be used for the sake of being "modern" but only when it serves to convey meaning more accurately and more succinctly than could be done by other means.
6. Individual differences in ability and motivation must be recognized even among college-capable students. Some material must be included for the student who has exceptional ability in mathematics.
7. Every effort should be made to acquaint the pupil with a variety of applications of mathematics to other fields. However, it must be recognized that valid application of school mathematics, suitable for inclusion in a text, are hard to find. They must not be too contrived, too trivial or too remote from the pupils' experience to be meaningful. For this reason it is probably better to provide applications by means of pamphlets which can be studied as supplementary materials by pupils who have the interest, experience, and scientific background necessary to profit from such a study.

Content and Organization

It was soon apparent that the content of the text would offer no serious problem. The teachers in the group reported that the eleventh grade program in most schools consists of a semester of algebra followed by a semester of either solid geometry or trigonometry and that a few schools offer a full year of algebra. The year of algebra was rejected in favor of including the essentials of trigonometry along with an introduction to vectors. Also, it was known that the SMSG tenth year geometry would contain considerable space geometry. For this reason, demonstrative solid geometry was eliminated from the eleventh grade outline. The algebra-trigonometry sequence was adopted as the most appropriate for grade eleven with the provision that graphical methods and coordinate geometry would be used where appropriate. The list of chapter headings developed during the initial sessions does not differ substantially from the one found in the revised edition. The fact that this sequence is already familiar to many teachers will no doubt serve to minimize objections to the use of this text. Moreover, teachers who agree with the recommendations of the Commission on Mathematics* will be pleased to see that the topics chosen for the eleventh grade are essentially the same as those recommended by the Commission.

The organization of this material was more difficult. One controlling consideration was the desire to advance the pupils' understanding of number systems. As the writing progressed this consideration began to permeate the entire text. Its main bearings will be found in Chapter 1 (Number Systems), Chapter 5 (Complex Number System), Chapter 12 (Polar Form of Complex Numbers) and Chapter 15 (Algebraic Structures). The development of the function concept can be considered a secondary theme in this text.

The writing group was much concerned about achieving effective coordination with SMSG texts for grades 9 and 10. While the group considered that coordination with the first course in algebra was essential, it also tried to write a text that would be usable by pupils who have not had previous SMSG courses. It is primarily for the benefit of such pupils that the rather extensive first chapter (Number Systems) is provided.

Modus Operandi

The group acted on the premise that the eleventh grade college preparatory pupils can read, (or at least can be taught to read) mathematical expositions. Although illustrative examples abound the pupil must read the text in order to do the exercises or pass the tests provided. Thus the group seeks to advance the pupil's ability to read the more advanced mathematics he will encounter in subsequent courses.

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* Report of the Commission on Mathematics of the College Entrance Examination Board "Program for College Preparatory Mathematics" 1959.

After the master outline had been adopted, the chapter became the basic writing assignment. After the general approach and basic content of a chapter had been considered by the entire group the chapter and its related Teachers' Commentary was assigned to a writing team composed of a mathematician and a teacher or in some cases, two teachers. While procedures varied a good deal, a common practice was for the mathematician to write the exposition while the teacher worked on exercises and made notes for the Teachers' Commentary. When a chapter was completed it was submitted to the group chairman who, with the advice of a small steering committee, accepted it or returned it for revision or rewriting (one chapter was rewritten five times). One of the principal functions of the chairman and his steering committee was to achieve coordination between chapters as well as between the eleventh grade text and other texts in the SMSG sequence.

Revision

The revision of the original (1959) version of the eleventh grade text was based on a careful study of the suggestions and evaluations which were submitted by the teachers who taught this material in the experimental centers during 1959-1960 academic year. In a very real sense, these teachers collaborated with the authors in an effort to make this text a more effective instrument of instruction.

Evaluation

Intermediate Mathematics will be evaluated by reviewers, publishers and by the increasing segment of the mathematical community which is interested in school mathematics. But, most important of all, it will be evaluated in the classroom. It is evident that new instruments must be developed for its proper evaluation. Many believe that manipulative skills will be retained longer and used more intelligently when they have been tied to the mathematical principles which give them validity. Nevertheless, a test which is primarily a measure of manipulative skills will not give a true picture of the concepts and insights which a pupil should gain from a study of this text. The group believes that this text, like other SMSG texts, conveys new dimensions of understanding and new attitudes toward mathematics which are beyond the range of conventional tests.

It is understood that new tests are being designed for the purpose of measuring the extent to which these alleged new dimensions of understanding have actually been acquired by the pupils. In the long run the success of this text depends on the extent to which it encourages emulation by commercial publishers and the extent to which it stimulates the students' interest and influences him to continue his study of mathematics in high school and subsequently in college.

A HISTORY OF THE 12TH GRADE SMSG

During the summer of 1958 at Yale, our group quickly decided to follow the general recommendations of the Commission on Mathematics. Accordingly we planned to write a text on Elementary Functions for a first-semester course, with the possibility of including sufficient material to serve as a full-year course for less adequately prepared students. Since the "gray book" already implemented alternative 1, Introductory Probability and Statistics, for the second semester, we decided to prepare a text for alternative 2, Introduction to Modern Algebra.

There was considerable discussion of the role of calculus and analytic geometry. It was generally recognized that it was likely that calculus would ultimately become the standard course in the 12th grade. There was a division of opinion on the extent to which a course in calculus should be pushed at this time. It was recognized that a fair amount of analytic geometry would be incorporated into the work of the previous grades. It was also contemplated that the Introduction to Modern Algebra would include some analytic geometry. Since in any case there existed reasonably satisfactory texts in calculus and analytic geometry and since there was considerable difference of opinion on the place of these subjects in secondary school, it was decided to address ourselves to the task of filling the most pressing textbook gaps.

The next step was to break the material into manageable blocks on which we could work in pairs to produce full outlines and (hopefully) first drafts. Priority was assigned to Elementary Functions. We were greatly aided by the Commission's outline which called for treatments of Sets, Relations and Functions, Polynomial Functions, Exponential and Logarithmic Functions, and Circular Functions. We omitted concern with permutations, combinations and mathematical induction which were covered in the gray book appendices. Since a rought draft on Polynomial Functions had been made for the Commission, we deferred work in this area. After the first week, we divided into two-person teams to work on the following topics: Sets, Relations, Functions, Exponentials and Logarithms, Circular Functions, Matrices, as a start toward the algebra course.

There were general meetings from time to time.

We produced (1) An elementary treatment of Sets, Relations and Functions (incomplete) with exercises. (2) A chapter on Exponentials and Logarithms in fairly finished form. (3) A full outline of Circular Functions from the point of view of the winding function. (4) A full outline of Matrices with typical examples and exercises, and a discursive treatment of part of the material.

Much discussion was devoted to the problem of overlap with the 11th grade, particularly with regard to trigonometry. It was agreed that we should not include the solution of triangles. However, no satisfactory resolution was found for the problem of treating the addition formulas and their consequences. The 11th grade wished to include them, since the mathematical training of many students would go no further. On the other hand, it would be absurd to discuss circular functions without the addition formulas. Moreover, we could not count upon knowledge of the 11th grade material on the part of the users of our text. The 11th grade wished to base the proof on the distance formula. One of us advocated a derivation based on complex numbers to avoid duplication and also because, in his view, it was neater and more natural. It was felt by many that teachers would shy away from complex numbers. Some of our group were also quite happy with the Cauchy proof. On the whole, therefore, it was agreed that, for the present, the overlap of content and method would be accepted.

There was some writing activity during the winter of 1958-1959. Sets, Relations and Functions, and Exponentials and Logarithms were revised as was the outline of Circular Functions. A draft of an alternative complex number treatment was written. Some revision of the Commission draft on polynomials, including a new treatment of synthetic substitution and continuity, was prepared, as was a non-calculus treatment of tangents to polynomial graphs and a set of maximum, minimum problems whose solutions were obtainable by these techniques. The Matrix draft was revised and additional material on Groups and Fields was prepared. Meanwhile, one of us wrote an alternative outline of an Introduction to Modern Algebra (including Number Theory).

Preparatory to the summer meeting at Boulder, the enlarged writing group was sent copies of such material as was available, as well as tentative timetables for the production of texts and commentaries, and tentative team assignments.

The 1959 writing group consisted of seventeen individuals, as compared with seven in 1958, including six who were present for periods ranging from one to six weeks.

At the outset of the session, certain policy questions were raised:

(1) Since the sets material was not used very much in the subsequent text, should it be substantially curtailed?

(2) Should the proposed treatment of tangents be adopted? If so, should expansions in powers of x-h be obtained by synthetic division or by the binomial theorem?

(3) How should the circular functions be treated?

(4) What should be done about the Modern Algebra course?

The following decisions were made:

(1) To abbreviate the treatment of sets somewhat, but keep most of it during this transition period. Subsequently, the treatment of functions was considerably modified, favoring mapping over ordered pairs on the ground that this is the way in which most mathematicians actually think of functions.

(2) It was agreed that the tangent approach based on the idea of linear approximation should be worked out. The binomial expansion method was favored partly perhaps because synthetic division would not be explained until the succeeding chapter on the algebra of polynomials, a chapter not yet written. This was a mistake which was remedied in the revised edition when the order of chapters was reversed.

Some of the group favored the calculus method over one which would be discarded in the students' later experience. The argument for the alternative approach was that in the time available for teaching tangents, it was difficult to avoid dangerous misconceptions or to give the student the impression that he knew more than he did and thus handicap the teaching of calculus itself. The merits of these arguments are still a matter for debate.

(3) A treatment of circular functions was devised which featured the idea of periodicity. The language of the winding function was not used. Stress was placed instead on the idea of rotation. After much discussion, a proof of the addition formulas was given in terms of vectors and rotations which was in the spirit of the previous discussion, instead of the distance formula method of the 11th grade. Reference to complex numbers was made only incidentally.

(4) It was decided to replace the course on Modern Algebra, outlined in the Commission's report, by one in Matrix Algebra on the ground that the matrix algebra course maintained contact with the concrete and practical problem of solving linear equations. Furthermore, the algebra of matrices leads more easily and naturally into a discussion of algebraic structure (rings, groups, fields) than the course previously outlined.

This decision was a difficult one. An outline of a very attractive alternative course had been developed which it might be useful to work out in detail. The plan was to abstract the concept of field from numerous concrete examples, and then to go on to finite fields with integers mod p, "discovering" Wilson's and Fermat's theorems in exercises. The next abstractions were to be Abelian groups, transformation groups and finally groups in general, culminating in Lagrange's theorem and a proof of Fermat's theorem, previously discovered. Numerous examples were to be included. Although we believe that our decision was the right one, a course along these lines would be of great value.

We organized into two subgroups, one concerned with Elementary Functions, the other with Matrix Algebra. Each of us read and criticized most of the material as it was produced. This work was invaluable in clarifying the treatment and stimulating incessant revisions. Although there was considerable crossfertilization, the general procedure was to assign each chapter to a small team of two to four members.

During the fall semester, work on the T. C. and on Chapters 4 and 5 was continued.

The summer was a very trying one. We were caught between the necessity of meeting deadlines and the desire to include a multitude of fresh ideas. The new members of our group contributed many valuable concepts which we tried as best we could to incorporate into a structure that perforce had to be broken into pieces and consecutively sent out of our hands. It was difficult indeed to steer a course with which we could be satisfied, but we did achieve a reasonable approximation to this goal.

One expedient was to place new material in the appendices. Examples are the work on mathematical induction and the treatment of area. An approach to exponentials and logarithms by way of functional equations was thought to be ahead of the times but will, we believe, come into its own.

The 1960 group at Stanford consisted of six individuals. We were assisted in checking exercises and the like by two high school teachers from the neighborhood.

In May, a small steering committee met in New York to consider the nature of the revisions to be made in the light of teaching experience and their own reflection. The role of calculus was again considered. The principal decisions reached were the following.

Elementary Functions

1. Reduce Sets to a brief appendix, omit Relations and treat Functions as mappings. Bring forward material on constant and linear functions, and composite and inverse functions.
2. Reverse the order of Chapters 2 and 3, placing the algebra of polynomials before the treatment of tangents. Omit the summation notation in the algebra chapter and generally simplify it. Place Newton's method in the new Chapter 3 after the treatment of tangents. The new arrangement had the advantage that the importance of

finding the positions of horizontal tangents for graphing could be brought out in advance.

3. The text should adopt neither the notation of the calculus nor the usual treatment of the tangent to a curve.
4. Tentatively, the treatment of tangents in terms of the wedge and dominance ideas should be accepted.
5. Simplify the rotation method of treating the addition formulas by using fixed vectors and an easier notation.

Matrix Algebra

1. Simplify Chapter 3 by using row operations directly and postponing elementary matrices.
2. Makes Chapters 4 and 5 more geometric.

This orientation proved to be invaluable, giving a strong sense of direction to our later efforts.

It proved possible to extend the wedge method to the tangents to exponential and trigonometric graphs and to generalize the idea to handle polynomial approximations of these functions. In the course of this development we eliminated all infinite processes and all explicit mention of limits.

The adopting of the new organization gave greater coherence and clarity to the text. By the addition of appendices on the solution of triangles, and on identities and equations, the treatment of circular functions became more nearly self-contained.

There were many improvements in detail; e.g., the careful distinction between polynomials and polynomial functions. Hardly a page remained untouched. The comments received from teachers were of course very helpful in eliminating errors and infelicities of expression.

Both texts differ somewhat from the recommendations of the Commission but not, it is felt, in spirit. The most important difference in Elementary Functions is the smaller role given to sets as a separate topic, the disappearance of the ordered pair definition of function in favor of the treatment of functions as mappings, and the elimination of all infinite processes. (See the treatment of the polynomial approximations for e^x , $\ln x$, sin x and cos x.) Permutations and combinations never got into the text and mathematical induction only in an appendix. Areas are treated in an appendix, a development not contemplated by the Commission. We believe that Elementary Functions provides a sound basis for the study of calculus.

Matrix Algebra differs from the Commission's recommendation but we believe that our solution is both teachable and sound. In particular, Chapter 2 on 2x2 matrices is an excellent introduction to abstract algebra. The Appendix - Research Exercises is a notable feature of the work. The geometry of Chapters 4 and 5 was somewhat streamlined and simplified. It is suggested that any future rewrite should give greater emphasis to vectors and analytic geometry. Matrix Algebra is a significant contribution to the teaching of mathematics.

History and Philosophy of the Writing Team for Grades 4, 5, and 6

Planning Conference, March, 1960; and Statement of Aims

Early in 1960 the Advisory Panel recommended that the efforts of SMSG in the improvement of the mathematics program be extended to include mathematics in the elementary school and that grades 4, 5, and 6 be the area to be considered first.

Accordingly a group of eleven persons* met for approximately one week in Chicago in March, 1960 to prepare outlines for a complete fourth grade text and to select units for grades five and six. It was planned, and subsequently carried out that these materials were to be written by a writing team during the summer of 1960. During the school years 1960-1961, 1961-1962 they were to be given try-outs with a large number of pupils and then revised and extended during the summers of 1961 and 1962.

The task of producing the outlines in March, 1960 was not an easy one and it was emphasized that the outlines which were produced were quite tentative, particularly concerning details, and that the writing teams would undoubtedly find it necessary to make many changes.

It was obvious that the materials to be written for grades four, five, and six should connect skillfully with those already written for the seventh grade as well as provide proper, adequate, and comprehensible mathematics for the pupils of the three grades involved. Consequently the outlines reflected the expectation that the writing teams would emphasize concepts, arithmetic and algebraic structure, derivation of algorisms from structure, problem solving, manipulative skills, proper use of simple mathematical language and symbols, and simple intuitive geometry. Although the deliberations that produced the outlines resulted in no precise statement of the comprehensive attitude to be adopted by the writing teams, it was felt that the objectives mentioned above were desirable, realistic, and compatible with the mathematics to be studied later.

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* Mildred Cole, J. A. Cooley, E. Glenadine Gibb, W. T. Guy, Jr., Stanley Jackson, Lenore John, John R. Mayor, Irene Sauble, Helen Schneider, Marshall Stone, J. Fred Weaver. Dr. Begle was present for the major part of the conference.

The mathematics to be written was to develop a wider range of ideas and a deeper insight into the fundamental concepts of mathematics than had been previously considered possible at this level. Little was known of the readiness and ability of the pupils to understand what was being contemplated. Comments from teachers in general indicate that we tend to underestimate what children can comprehend and do. Some concepts at first appear to be unteachable but are found on closer analysis and teaching experience to be appropriate for elementary school pupils. It was recognized that a major difficulty would be encountered in presenting ideas in precise language and that compromises would have to be made in language use in order to prevent the young pupil from losing sight of the central idea. However, no inconsistencies in terminology were to be used just to be removed later.

A part of the basic assumption was that the interest of the pupils must be obtained quickly and that the program should be built to maintain this interest. In order to capture interest in the beginning of the fourth grade it was decided that the fourth grade should not begin with a review of the arithmetic of the third grade. The decision was made that the fourth grade should be concerned with the fundamental operations with whole numbers, the concept of fraction, and simple intuitive geometry in two and three dimensions. The fifth grade outline called for considerable study of the operations on fractional numbers, different numeration systems, and extension of the study in geometry.

The outline for the sixth grade presented one problem which was not met so sharply in the outlines for grades four and five. This concerned the inclusion of some topics which had been placed in the seventh grade. Reports from experimental centers that had used seventh grade material suggested that certain topics could be taught successfully in grades earlier than the seventh and thus make room in the seventh grade for new topics or ones to be moved there from succeeding grades. The topics of Negative Numbers, Coordinates, Volume, Ratio, Graphs, and Exponents were accordingly placed in the sixth grade outline.

The persons who worked on the outline were aware that it was not perfect. The section on Content indicates in detail the nature of the outline or perhaps better, the outline as it eventually emerged from the writing sessions. Evidence that the outline was meant to be subject to revision is seen in some of the comments made after the planning conference and attached to the outline. " . . . teachers and public will need to be informed that although some of the skills are not mastered as early, others are mastered earlier." "I think the writing group should have instructions to work on graphing where it may seem most appropriate. Possibly the circular graphs could be brought in in connection with fractions in Grade 4, Chapter 9." "Guard against getting in the situation of not having new ideas in each grade." "Include something on limit, e.g., take a rectangle showing that as it gets nearer and nearer a square it gets larger." "Care must be taken that there shall be sufficient work on operations with measured quantities, i.e., hours, minutes; pints, quarts, gallons, etc., so that students will not be incompetent in these routine matters."

The outline went to the Elementary Writing Team for their first session in the summer of 1960.

Writing Sessions

The first elementary writing team convened at Stanford for eight weeks in the summer of 1960. The team of twenty-five persons consisted of elementary, junior high and senior high teachers, supervisors, specialists in mathematics education, and college and university mathematics teachers. The panel concerned with its supervision appointed a steering committee of four (a fifth was soon added) whose function was to organize the team into writing groups, make writing assignments, carry out the writing assignments it assigned to itself, keep the writing of the groups coordinated, arrange "hearings" and team discussions, and expedite all matters that would lead to an acceptable product in good time. The steering committee had one meeting two weeks prior to the beginning of the writing session and another on the day before the session began.

The mathematicians and consultants guided the writing team in its strong effort to base the content on sound mathematical principles and provide the teachers with the mathematical background needed to teach the material. The chapters in the Teachers Commentary were in one-to-one correspondence with the chapters in the pupil text. The teacher was provided with answers to questions and solutions to the problems in the pupil text and suggestions on procedure in teaching. Emphasis was placed on the guided discovery method of teaching and text material for the pupil was so written. Each chapter of the Teachers' Commentary provided a careful exposition of background mathematics required in the chapter with understanding and confidence. Suggestions to the teachers are not intended to restrict or stifle creative approaches of the teacher but only to make certain that the teachers clearly understand the essence of the ideas to be presented.

All persons on the team participated in the writing. The teachers as well as the mathematicians were sensitive to "good and bad" mathematics and were particularly alert to the teachability of the written product. All writers were aware, or soon became aware, of the necessity for clarity of exposition and appropriateness of vocabulary. No central ideas were to be obscured in a linguistic maze. The materials were to be mathematically sound, clear in language and symbols, sufficiently easy to read so that reading would be encouraged, and as appealing in appearance as possible.

The panel on elementary mathematics asked the elementary writing team to prepare the first chapter of the fourth grade (Concept of Sets) as quickly as possible after reaching Stanford in order to give it a try-out with fourth grade pupils. The steering committee prepared in one week such a unit and a young Palo Alto teacher taught it to one group of fourth grade children and to another group of sixth grade chidlren. As might be expected the sixth grade children required less time but all the young pupils were enthusiastically interested and at the end of the time gave good evidence of clear comprehension of what they had studied. This was an encouraging beginning for the writing session.

The writing team reached its quantitative goal by the end of eight weeks, the materials were printed, and distributed to the experimental schools. (It would require a volume of considerable size to describe all matters involved in caring for all the details involved in these processes.)

In October, 1960 the teachers who were to teach the experimental materials convened in Chicago with members of the writing team. Objectives of the program were discussed and clarified, the teachers were encouraged to approach their task with faith and without prejudice, and were urged to send to SMSG headquarters their full and frank appraisals of the materials relevant to a revision in the summer of 1961.

In summer 1961, the writing team met at Yale and carried out the first revision. The steering committee met a few days early to make preparations of assignments. The writing team required very little time to get into its job. The revisions consisted mainly of re-ordering some of the topics, finishing chapters either left incomplete or unattempted in the summer of 1960, and in changing the style of the books from consummable to non-consummable. Consummable books were designed to be used only once--non-consummable more than once. The session ended with fairly good satisfaction of the treatment of all topics except that of rational numbers and a second revision was predicted for it. Some incomplete units were taken by members of the steering committee to be completed by December 31. This was done.

In summer 1962, the writing team met again at Stanford. Since the requirements were not so large the team was decreased to twelve in number. The major revision was as predicted in the chapters involving rational numbers. It had been the practice in 1960 and 1961 to permute the members of the small writing groups but in 1962 the writing groups were not changed during the summer. Consequently one writing group devoted its entire time to the chapters which involved the rational numbers. Two persons were added to the writing team for the sole purpose of writing problems. Sets of problems were written for each chapter and for cumulative review. Two members of the steering committee devoted part time to the K-3 writing team which met for the first time in the summer of 1962.

The Advisory Panel accepted the 1962 revision of the elementary texts as the final one.

Content

The chapter headings in the two columns will give some impression of the differences between the original outline and the final version.

	Original Outline		Final Outline
	Fourth Grade		
1.	Concept of Sets	1.	Concept of Sets
2.	Numeration	2.	Numeration
3.	The Number Line	3.	Properties and Techniques of Addition and Subtraction I
4.	Techniques of Addition and Subtraction	4.	Properties of Multiplication and Division
5.	Point Sets	5.	Sets of Points
6.	Recognition of Common Figures	6.	Properties and Techniques of Addition and Subtraction II
7.	Properties of Multiplication and Division of Whole Numbers	7.	Techniques of Multiplication and Division
8.	Techniques of Multiplication and Division	8.	Recognition of Common Geometric Figures
9.	Developing the Concept of Fraction	9.	Linear Measurement
10.	Linear Measurement	10.	Concept of Rational Numbers

Fifth Grade

1. Different Names for the Same Number
2. Properties and Techniques of Addition and Subtraction of Fractions
3. Side and Angle Relationships of Triangles
4. Measurement of Angles
5. Extending Systems of Numeration
6. Multiplication using Decimal System of Numeration for Whole Numbers and Fractions
7. Area
8. Division using Decimal System of Numeration for Whole Numbers and Fractions

1. Extending Systems of Numerations
2. Factors and Primes
3. Extending Multiplication and Division I
4. Congruence of Common Geometric Figures
5. Extending Multiplication and Division II
6. Addition and Subtraction of Rational Numbers
7. Measurement of Angles
8. Area
9. Ratio
10. Review

Sixth Grade

1. Negative Numbers
2. Coordinates
3. Multiplication and Division of Fractions
4. Volume
5. Ratio applications, Graphs, Central Tendency
6. Exponents and Scientific Notation
7. Sets and Circles

1. Exponents
2. Multiplication of Rational Numbers
3. Side and Angle Relationships of Triangles
4. Introducing the Integers
5. Coordinates
6. Division of Rational Numbers
7. Volume
8. Organizing and Describing Data
9. Sets and Circles
10. Review

The usual topics presented in conventional courses for these three grades are covered but from new approaches and with greater depth. Early in grade 4 the emphasis is on helping children to understand the *nature* and *properties* of addition, subtraction, multiplication, and division as *operations of mathematics*. It was assumed that the children had had some experience with these operations, particularly addition and subtraction, but that this was the first time emphasis will have been given to the *nature* of these operations, their relationship to each other, and these properties: commutative property, closure property, associative property, distributive property of multiplication over addition, and the special properties of the numbers one and zero.

An operation is described simply as a way of thinking about two numbers to produce a third number. The use of the term "binary operation" with the children at this stage is left to the teachers' judgment but the idea that the operation is performed on just two numbers to produce a unique third number is made clear to them. Addition is presented as an operation on two addends to produce a third number called the sum. Subtraction is thought of as an operation for finding an unknown addend if the sum and the other addend are known. From the beginning the children are encouraged to relate subtraction to addition and are made aware of the inverse relationship. The number line and sets are used as models to help the children in "seeing" the meaning of addition. Multiplication is taught primarily as an operation in its own right with certain characteristic properties. Division is presented in terms of multiplication as an operation on two numbers, a factor and a product, to produce an unknown factor. Arrays are used throughout the work on multiplication as a physical model with which this operation is associated. They serve as an effective aid to understanding the nature of multiplication, its properties, and its inverse relationship to division. Extensive use is made of "mathematical sentences" to help the children sense the relationships between the operations and to decide what operation to use to find an unknown number. Basic to the understanding of the operations is an understanding of principles of numeration. An initial treatment of this, with emphasis on the decimal system, is presented in Chapter 2 of

Grade 4. Another chapter on numeration in Grade 5 seeks to extend and deepen the pupils' understanding of place value and the decimal system by exploring numeration systems with different bases. Only when the decimal system is studied in the context of place-value systems do certain of its properties emerge clearly. The simplest treatment of the operations and numeration is presented first with relatively small numbers. These ideas are re-examined, reenforced and extended in later chapters using larger numbers.

When symbols facilitate the presentation they are introduced and their use explained. When we write $5 + 2 = 8 - 1$ we mean that $5 + 2$ and $8 - 1$ are different names for the same number. When we write $\overline{AB} = \overline{CD}$, we mean that AB and CD are two names for the same line segment; that is, they name the same set of points. The children learn that mathematical symbols can be used as a means of communicating ideas and relationships.

It is of utmost importance to begin the development of an effective approach to problem solving in these early grades. No attempt is made to follow the usual step-by-step question and answer procedure used in many conventional texts. The person who has developed skill in problem solving not only has many plans but frequently uses his steps in a different order in solving different problems. If children are to become successful in problem solving they must learn to describe the problem situation by writing the mathematical symbolization for it. The developmental material on problem solving encourages the children to interpret the problem situation in a mathematical sentence, study the mathematical sentence and decide what operation to use to solve it. It is important that the children understand that the mathematical sentences they write are always about numbers. The problem may refer to books, stamps, dollars, or rockets, but the sentence is about numbers only. An answer sentence is written to use the result of solving the mathematical sentence to answer the question in the problem. The plan, as suggested, is simply a means for helping children to extract numbers from the action implied in the problem and record their thinking in a systematic manner. Eventually they learn to analyze and solve more complex problems and to express the structure of the situations by using several mathematical sentences to solve one problem and to combine several sentences into one sentence. This approach should build an excellent background for solving problems in algebra.

Approximately one-third of the materials in the three grades is geometry. Concepts from plane and solid geometry are presented together. The approach is intuitive and informal. Attention is directed to the development of such mathematical ideas as point, line, plane, space, angle, and simple closed curves. The children make and handle many of the physical models. Patterns for making some of the common surfaces such as the rectangular prism, pyramid, and cube are in the Teachers' Commentary. From these representations, the children observe the plane geometric figures that are parts of the solid figures. The notion of congruence is developed first in comparing line segments and extended to achieve familiarity with the intuitive concept of congruence as applied to triangles. Throughout the work there is emphasis on the fact that the models being used are just representations of sets of points in space. Some children readily understand this abstract notion. Others still require a point to have size and to be represented by physical objects. Their understanding will continue to undergo refinement as their progress in geometry continues.

There is a treatment of measurement throughout the sequence. Through the use of arbitrary units children are led to understand such basic concepts as the following:

(1) The measure of a geometric object (line segment, angle, plane region, solid region) in terms of a unit is the number (not necessarily a whole number) of times the unit will fit into the object.

(2) The meaning of a standard unit of measure and the need for standard units for purposes of communication.

(3) The unit must be of the same nature as the thing to be measured: a line segment as a unit for measuring line segments, an angle as a unit for measuring angles, etc.

(4) As the unit becomes smaller, the interval within which the approximate measurement may vary decreases in size. It is of interest to note that in the development of the ideas of measurement for the children, we are actually following the historical development of this concept. Although it is typical of all SMSG

material to give greater depth of a topic, nowhere is this more apparent than in the treatment of measurement.

Extensive use is made of sections in the children's book titled "Exploration," or "Thinking Together," or "Working Together." These sections design experiences that encourage the children to explore for themselves, to raise questions, and to investigate ideas. The material in these sections is sometimes written in such a way that the children reach conclusions by guided discovery. Through the use of skillful direction and questioning they are led naturally into a curiosity to seek answers to significant questions. There are sets of exercises for the children to work independently of the teacher, to apply generalizations, to promote their facility in mathematical manipulations, and deepen their understanding of the concept involved. The problem sets are carefully constructed, and range in type from simple drill exercises to very difficult items called "Braintwisters" which challenge the very best pupil.

The rational numbers, always a troublesome topic, are presented simply as an extension of the whole numbers. Three different kinds of physical models (regions, sets of objects, and the number line) are used to develop the idea of rational number and explain the operations. By partitioning regions and line segments into congruent parts and sets of objects into equivalent subsets, the idea of rational number is developed as an infinite set of ordered pairs of whole numbers where the second member of the set is not zero. The properties and operations of the whole numbers are re-examined to emphasize the common structure of the two systems and each is used to re-enforce the other. A fraction is defined as a symbol which names a rational number and a decimal a symbol which can name a rational number when the second member of the pair is 10 or a power of 10 . A pattern for matching fractions with points on the number line helps the children see that we match fractions such as $\frac{4}{4}$, $\frac{3}{3}$, $\frac{2}{2}$, $\frac{1}{1}$, with the same point and that this point is also matched with the counting number 1 . They observe that this occurs also with fractions such as $\frac{8}{4}$, $\frac{6}{3}$, $\frac{4}{2}$, and the counting number 2 and so on. Further investigation leads them to discover that every rational number is either a whole number or lies between two consecutive whole numbers. But unlike the whole numbers they can "see" that if they know one rational number they cannot name the next one. This pattern of matching also reveals that fractions such as $\frac{1}{2}$, $\frac{2}{4}$, $\frac{4}{8}$, $\frac{3}{6}$, name the same rational number and that like the whole numbers the same rational number has many names. They are led to inquire whether the familiar properties of the operations with whole numbers are true for the operations with the rational numbers, and deduce, for example, that addition and subtraction of rational numbers with the same denominators reduces to addition and subtraction of whole numbers and in fact all of the operations with rational numbers reduce to manipulations with whole numbers. The models used also provide the ideas necessary for problem solving with rational numbers.

Topics not usually included in these elementary grades such as factors and primes, the integers, coordinates, and exponents are studied in Grades 5 and 6. These chapters develop important fundamental mathematical ideas and build a valuable background for future study. As an added bonus, they also provide a new setting for the "maintenance" of skills previously taught.

The arrangement of topics is influenced both by necessary sequence and attempt at variety. The geometry chapters are placed at intervals throughout the texts to break up the presentation of arithmetical ideas for the sake of variety and a change of pace. Children who have had difficulty with the techniques of arithmetic frequently get a fresh start and show renewed interest and enthusiasm when they come to a geometry chapter. At the request of the teachers, practice exercises and review problems are placed throughout the texts. These are intended to be used as needed. Individual differences among children necessitate varying amounts of practice and review in the development and maintenance of ideas and skills.

The best means of determining the intentions of the writers is an inspection of the contents of the books. The elementary writing team hopes that such an inspection will reveal that the content presents a wider range of ideas than is usually found in elementary texts and that there will be greater understanding of the basic concepts. Also that there is a gradual revelation to the pupil of the deductive nature of mathematics, emphasis on making generalizations, and that relationships among the various topics have been exploited so that knowledge of one contributes to understanding the others.

Report on the Preparation
of
INTRODUCTION TO SECONDARY SCHOOL MATHEMATICS

by

Mildred Keiffer

The Introduction to Secondary School Mathematics is adapted from Mathematics for Junior High School - Volumes I and II. It was prepared on the recommendation of the Panel on Underdeveloped Mathematical Talent which met initially in August, 1959, to consider the problem of attracting more students into the study of mathematics and of retaining them for more years than was currently the case. During the preceding summer the sample text materials prepared for grades 9 through 12 had been written for the group of able students from which future leaders in science and mathematics are reasonably expected to come. The aim in the production of the seventh and eighth grade materials was broader than this, since the study of mathematics is required of all students in these grades.

Mathematics for Pupils Not in the Upper 25 Per Cent

It seemed appropriate to the Panel to give attention to the kind of mathematics the average student should have with the thought that he should have as much as possible. In this group of students not included in the gifted group, but still very possibly anticipating college, there may well be undiscovered and undeveloped capacity for profitable study of more mathematics than heretofore had been supposed.

If the content recommended for Mathematics for the Junior High School is necessary for the ablest students, it is reasonable to suppose that these basic ideas are equally important for other students as well. The Panel proposed that certain pilot classes be asked to try the SMSG seventh grade materials in order to secure information about reading level, inherent motivation, and content appeal. Reports from the use of selected units indicated that interest was high. It was difficult to decide whether the reading level was too demanding or whether these students were simply not accustomed to reading mathematics.

Instructions for the Writing Team

It was agreed that writing teams should be asked to revise the SMSG texts for the 7th and 9th grades, taking into account the problems of motivation, reading level, and interest span but retaining the basic structure of the courses. These revised materials were then to be used to test the hypothesis that average students can learn the kind of mathematics in the standard SMSG textbooks provided that they are encouraged to proceed at their own pace through a presentation especially designed for them.

Modification and Revision Plans

During the summer of 1960 a team of teachers and mathematicians revised Mathematics for Junior High School - Volume I, Parts 1 and 2. The changes made were designed to simplify the presentation and to reduce the reading difficulty. The writers were assisted by a professional analysis of the readability of Mathematics for Junior High School. They also studied the comments and reports of the teachers who had used selected units in their classrooms.

In some places a reduction in the length of the explanatory sections was made and more exercises were included to lead the student in simple steps to appropriate conclusions. Frequent short lists of problems were preferred to less frequent long lists. An attempt was made to increase participation by the student in the development of the successive ideas. Discovery by means of "experiments" was extended in the geometry sections.

In rare cases, concepts regarded as too demanding were postponed, but for the most part the writers retained the same content. Sentences were shortened and the vocabulary was simplified as much as possible. Abstract principles were introduced by means of specific concrete examples. The appearance of the page was a matter of concern and an attempt was made to reduce the forbidding aspect of solid pages of reading matter.

The intent of the writers was to retain essentially the same subject matter but to prepare it specifically for the student in junior high school whose mathematical talent remained underdeveloped. The writers were constantly aware of the importance of this body of mathematics for all informed citizens in our society as well as for the pre-college student as he prepares for advanced work in related subjects. It was the hope of the writing team that this material would serve to awaken the interest of a large group of junior high school students whose ability to learn mathematics had not been recognized or whose progress had been blocked in an inappropriate curriculum.

During the summer of 1960 the writing team completed 15 chapters to form Parts 1, 2, and 3 of Introduction to Secondary School Mathematics. The original chapters on Mathematical Systems and Mathematics at Work in Science were omitted. Certain other chapters of the first SMSG textbook for grade seven were separated into 2 chapters each and the order was slightly altered. The original Chapter 4, Non-Metric Geometry became Chapter 4, Non-Metric Geometry I and Chapter 7, Non-Metric Geometry II. Chapter 6, the Rational Number System, became Chapter 6, Rational Numbers and Fractions, and Chapter 8, Rational Numbers and the Number Line. Chapter 9, Ratio, Percent, and Decimals became Chapter 9, Decimals, and Chapter 10, Ratio and Percent.

Organization and Procedure

The writing group functioned as a team throughout. Each writer was responsible for the first draft of a specific chapter which was then reviewed by the whole group. Since exercises were used as much as possible to contribute to the development of a concept, exercise sets were written as part of the chapter. Before any work was begun on any chapter, the whole group discussed the changes which should be made. One mathematician in the group accepted the responsibility for reading and approving every chapter to make sure that the mathematics was correct. Two other members read every chapter for general editing purposes after all correct mathematics and appropriate presentation matters had been attended to. Each writer also prepared the Teacher's Commentary to accompany his chapter as he worked on the student text. The advice to the teacher about how to handle the presentation was considered of prime importance. It was held equally important that the Commentary help the teacher extend his command of the subject matter beyond what he taught to his students.

During the following summer at Yale (1961) an expanded team made minor changes in most of Parts 1 and 2 as recommended in the teacher reports, and wrote Part 4. Few reports were submitted on Part 3 since few classes had time to use it. Hence revision of these chapters was deferred until the next summer.

The Panel continued to be concerned with tests and how to find answers to such questions as: Is there a segment of the secondary school population below the top 25 per cent which can master the mathematics recommended for the ablest group if more time is given to the undertaking and the presentation is designed for the group in question? If this answer is yes, how much time is needed? How can we determine whether the Introduction to Secondary School Mathematics actually teaches students the content in the standard SMSG junior high school mathematics (Volume I)?

The Panel recommended that Part 4 be based upon selected topics from the standard Junior High School Mathematics - Volume II for the eighth grade. The selected topics were to be those which would be most helpful in the preparation of the students for their subsequent study of algebra. The chapters which were prepared, with the accompanying Commentary for Teachers, were: Operations with Rational Numbers; Equations and Inequalities; Coordinates in the Plane and the Pythagorean Property, Real Numbers; and Scientific Notation, Decimals, and the Metric System. The chapter on Rational Numbers and Fractions continued to be one of the most difficult for teachers to present and students to follow, so additional effort was made to improve it.

Final Revision

The final writing session was held at Stanford in 1962. Six writers including one hearty mathematician, prepared a final revision of all 4 parts of the Introduction to Secondary School Mathematics. Preliminary results from the Minnesota experiment indicated that the modified (Introduction) text and the standard seventh grade SMSG text taught the same mathematics, as measured by student scores on the SMSG block tests. On the basis of experiences reported by the Centers it appeared that the hypothesis had been substantially confirmed that an appreciable number of seventh grade students of average and even lower than average ability could learn the mathematics in the standard SMSG seventh grade text if it were presented in the format of the modified text. In addition, the modified seventh grade text had been used successfully with slower ninth grade students and in general mathematics classes at the tenth, eleventh, and twelfth grade levels.

Before the final revision was started, the teacher reports were studied with great care and used extensively in the pre-planning by the total group. Work on the first five chapters was completely re-written in a last attempt to answer the pleas for help. Rational Numbers and the Number Line received careful revision and another chapter was subdivided into Chapter 13, Angles and Parallels and Chapter 14, Polygons and Prisms. Part 4 received the same careful attention. All 5 chapters were completely revised in the light of the comments from the classroom teachers. Again, the total group reviewed every re-written section for the text and the Commentary and the mathematician member read and approved all the material. Two members of the team read every chapter for a final check. Because of the many meetings required for the group discussions both before work on a chapter was begun and after each successive draft was prepared, individual production required midnight and week-end time. Like other SMSG writers, those who worked on the Introduction to Secondary School Mathematics were pleased to be of help in providing for the improvement of mathematics instruction for increased numbers of students.

The final printing of the Introduction to Secondary School Mathematics is in two volumes. Volume I is in 2 parts and includes 5 chapters in each part. Volume II has 6 chapters in Part 1 and 5 chapters in Part 2. Because of this publication plan and because of the non-graded title, the material in part or as a whole, may be used anywhere from grade 7 to 12 where it can serve to good purpose.

Report on the Preparation

of

INTRODUCTION TO ALGEBRA

by

Walter Fleming

The Introduction to Algebra was prepared on the recommendation of the Panel on Underdeveloped Mathematical Talent which had its first meeting in August, 1959. A preliminary edition of SMSG's regular 9th grade text, First Course in Algebra, intended for students of high ability, had been prepared that summer at Boulder. The panel gave its attention to the mathematics which an average ninth grade student should have. It was recognized that in this group of 'average' students there are many able students whose mathematical talent has not been discovered. The following question was considered: If SMSG were to prepare a sample text for the average ninth grade student, what mathematics should it contain? The answer arrived at was that it would be desirable if the content would be essentially that of SMSG's regular ninth grade text. It was anticipated that it would take an average class longer than a year, perhaps two years, to cover all of the material in the proposed text.

It seemed worthwhile to the panel that SMSG should test the following hypothesis: A student of average ability can learn the same kind of mathematics as that which is contained in SMSG's First Course in Algebra, provided that the material is presented in a way especially designed for him and provided that he is encouraged to proceed at a pace appropriate for him. Out of these deliberations came the recommendation that a writing team should be selected for the preparation for this new text.

The Work of the Writing Team during the Summer of 1960

A writing team, consisting of 8 teachers and 4 mathematicians, began work on the new project at Stanford during the summer of of 1960. The team was called the 9-M group (to distinguish it from the 9-F group which was also at Stanford that summer working on the final revision of the regular ninth grade text). The mandate given to the team by the Panel on Underdeveloped Mathematical Talent seemed clear enough, namely, that it should write an algebra book for average students to parallel the regular ninth grade text. Before actual writing could begin the writers had to become thoroughly familiar with the content of the regular ninth grade text. Frequent conferences with members of the 9-F team proved to be very profitable. It was also necessary that substantial agreement should be reached concerning the way in which the 9-M writers would seek to carry out their work. There was much discussion on the subject of the level of difficulty which the new text could afford to have. As a result of all this it turned out that no actual writing was done during the first two weeks.

During those first two weeks of discussion and contemplation, the writers agreed on the following aims and objectives which they would seek to carry out in their preparation of the text, Introduction to Algebra:

(1) The vocabulary should be at about the 5th grade reading level.
(2) New concepts should be introduced one at a time and through concrete examples.
(3) Expository sections should be kept as short as possible.
(4) Numerous illustrative examples should be used.
(5) There should be an adequate supply of drill material.
(6) There should be sets of cumulative review problems.
(7) A summary or review should appear at the end of each chapter.
(8) There should be a cumulative index and a glossary of terms at the end of each of the four parts of the text.

The material produced that summer appeared in three parts. The first two parts paralleled the first half of the regular 9th grade text both in content and structure. It was thought that this might be enough material for a full year's course for a class of average students. The third part consisted of a miscellany of topics from the second half of the regular 9th grade text.

In preparing the text Introduction to Algebra the team used a division of labor by function rather than by chapter. First drafts were written by various persons, but two of the writers wrote all final drafts. A full-group hearing was held on each first draft, however, before the final draft was written. A special exercise writing team was established, and the exercises for each chapter were prepared as soon as the first drafts of the expository material were completed. The text material in the teachers' commentary was prepared by whoever wrote the chapter. The chairman of the group reviewed each chapter before it went to the typist.

Completion and Revision

During the summer of 1961 the 9-M writing team (its membership altered somewhat) continued its work, this time at Yale University. Teachers' reports sent in from various try-out centers where the preliminary edition, Parts 1 and 2, had been put to classroom use, were studied by all members of the writing team. In the revision of Parts 1 and 2 careful consideration was given to the suggestions and criticisms contained in these reports. Parts 3 and 4 were written (the old Part 3 was scrapped) to parallel the second half of the regular ninth grade text. The material in Parts 3 and 4, it was thought, would constitute a full year course for classes of the kind for whom the text was designed. Unit tests were prepared covering the material in all four parts of the text.

The final revision of the entire text was accomplished by the team (reduced in size to six members) during the summer of 1962 at Stanford. Once again all available reports from teachers were studied and many of their suggestions were utilized. Unfortunately, reports on Parts 3 and 4 were almost non-existent, since this material had been tried out in only a handful of classrooms. Revision of this portion of the text had to proceed on the basis of the writers' wisdom and good judgment. Much attention was given to the improvement of the quality of the problems and exercises. In many of the chapters the text material in the teachers' commentary was rewritten to give the teacher a better idea of how the material in the particular chapter ties in with what has gone on before and how it foreshadows what is to come in later chapters.

Some Noteworthy Characteristics of the Text

The writers always kept in mind that they were writing an algebra text for students of average ability. They knew that they had to take into account problems of motivation, reading level and interest span. At the same time, the basic structure of the regular 9th grade text was to be preserved. These concerns on the part of the writers are reflected throughout the text.

Some noteworthy characteristics of the text, Introduction to Algebra, are given below. They serve to illustrate in what ways the writers sought to achieve the aims and objectives which they set for themselves at the beginning of the project.

(1) The material is expanded to 18 chapters (this corresponds to 14 chapters in the First Course in Algebra). If each chapter had been made to correspond to a chapter in the regular text, several of the chapters in Introduction to Algebra would have become excessively long.

(2) Many of the expository sections are interspersed with oral exercises and problem sets. The thought behind this is that the average student cannot assimilate large amounts of new material at one time. It is better to present him with one concept at a time and to give him some experience with this concept through problems and exercises.

(3) Various devices, some of them quite artificial, are used to make the printed page more readable. Sentences are kept quite short, and free use is made of rhetorical questions. Important things are made to stand out by enclosing them in boxes, or by the use of generous spacing and indentation.

(4) The spiral technique is employed in the presentation of concepts which are particularly difficult for students to grasp and which have repeated applications in the text. A good example of this is the way in which the distributive property is handled. It appears the first time in the form of a simple numerical example.

Every time it reappears it gains strength and the student's grasp of the idea should become firmer.

(5) The summary at the end of each chapter provides the student with a convenient and conspicuous survey of the important ideas contained in the chapter. Review is especially important for the slower student.

(6) The text contains a much greater supply of the easier type of exercises and problems than are found in the regular 9th grade text. The writers were convinced that the average student needs these easier assignments before he can proceed to the more difficult ones.

(7) At the end of each chapter there is a review problem set. These review problems serve an important purpose for the student, especially the less gifted student. They help to consolidate learning so that a better foundation is laid for the lessons ahead.

(8) Proofs appear in this text but in less abundance than in the First Course in Algebra. It was felt by the writers that it would have been a mistake to include a large number of proofs. The sheer bulk of them would tend to discourage the student. A real attempt is made to get the student to understand and appreciate what a good mathematical proof is like. It is good for him to see how the properties which he has studied are used in the construction of a proof.

(9) The discussion of systems of open sentences in Chapter 17 presents quite a different approach from that which is used in the regular 9th grade text. The writers exploit to the fullest extent possible the notion of equivalence of compound sentences. This makes for a unified treatment of the various types of systems of open sentences. Teachers have commented that students appreciate this treatment.

The writers of SMSG's Introduction to Algebra hope that this text will make a significant contribution toward attracting more students into the study of mathematics and give them the kind of background in the subject which will enable them to continue their study in subsequent years.

A HISTORY AND PHILOSOPHY OF GEOMETRY WITH COORDINATES

by

Lawrence Ringenberg

I. HISTORY

Princeton. In the spring of 1961 a group of mathematicians and mathematics teachers met for one week at Princeton University to plan a new SMSG textbook writing project. Present at that meeting were Mr. James Brown, Atlanta, Georgia; Miss Janet Coffman, Baltimore, Maryland; Professor Copeland, University of Michigan; Mr. Eugene Ferguson, Newtonville, Massachusetts; Professor Good, University of Maryland; Professor Levi, Columbia University; Professor Ringenberg, Eastern Illinois University; Professor Rosenbaum, Wesleyan University; Mr. Harry Sitomer, New York City; Professor Spencer, Williams College; Professor Tucker, Princeton University; Professor Wylie, University of Utah. Dr. Begle, Director of SMSG, was present for the first day, and his assistant, Dr. John Wagner, was present for the last day of these planning sessions.

The Commission on Mathematics in its statements on secondary school geometry courses included recommendations (1) that plane and solid geometry be developed as an integrated course, (2) that coordinates be introduced into the course early, (3) that coordinates be used in the proofs of theorems whenever it appeared easier to do it that way, and (4) that vectors be introduced and used whenever it appeared easier to use them.

The SMSG geometry project which started in 1958 was well received; the next text produced by this project was considered by a great majority of the mathematical community to be a success of a high order. By 1961 many schools were using this text; many college courses for high school teachers, including courses designed as part of several NSF summer institute programs for teachers, were concerned with the mathematical and pedagogical implications of the SMSG geometry. The history and philosophy of the SMSG (regular) geometry appears elsewhere in these reports.

For the purpose of this present report it is sufficient to note that the SMSG (regular) geometry course does not include vectors, and that coordinate geometry is introduced in the last chapter of the text.

At the meeting in Princeton we were told that the SMSG panel on textbooks had directed that another geometry course be developed, one which would introduce coordinates early and use them frequently throughout the course, and which would introduce vectors and vector methods. As in all SMSG textbook writing projects, the panel undoubtedly recognized the experimental nature of this project. It was a project which was not likely to be undertaken for financial gain. It was a project which needed to be done if we were to learn the feasibility of exploiting coordinates and vectors in high school geometry. It was a project which required the cooperative efforts of a group of people with a variety of backgrounds. It was a project which needed financial support. In short it was the type of project for which SMSG exists.

A considerable portion of the Princeton session was spent in discussing basic issues. One issue was concerned with the proper level and place of rigor in a text for tenth graders. How formal should a good mathematics text be? Can we develop mathematics deductively without a formal postulational basis? What background in terms of set language and theory and real number theory shall we assume? Shall we summarize this background without indicating its formal development? How shall we introduce distance? coordinates? How shall we treat congruence and similarity? Shall we mention non-Euclidean geometry? How important are two-column proofs? How does one cite authority for a step in a two-column proof if that authority is in the algebra or arithmetic which is assumed rather than in the formal geometry structure being developed? To what extent shall we use paragraph-style proofs? Shall we teach the students to write paragraph-style proofs? Shall we integrate plane and solid geometry from the beginning or wait until after triangle congruences have been treated? Shall we treat distance and angle measure based on one unit for each, or shall we develop a mathematical basis for measure relative to a unit? Shall we assume the parallel postulate? Should we defer angle measure until late in the course, using an affine geometry development? How shall vectors be introduced? What examples shall we use to illustrate the power and beauty of vectors?

The latter part of the week at Princeton was spent in developing a list of chapter titles and some tentative outlines. The discussions regarding the use of an affine geometry approach led to a resolution which was passed unanimously by the Princeton team. We recommended to SMSG that a pilot experiment be conducted to explore the possibility of an affine geometry introduction for high school formal geometry.

Yale. The geometry writing team at Yale for eight weeks in the summer of 1961 included all of the Princeton group (with the exception of Professor Tucker) and, in addition, the following persons: Mrs. Virginia Mashin, San Diego, California; Miss Cecil McCarter, Omaha, Nebraska; Mr. John Murphy, San Diego, California; Mr. William Oberle, Baltimore, Maryland; and Mrs. Laura Scott, Portland, Oregon.

Mr. Sitomer, Professor Spencer and Professor Wylie had written independently drafts of Chapter 1 prior to the beginning of the Yale session. These drafts served as a point of departure in developing a style or pattern for the text. We divided ourselves into several smaller teams: leader and liaison, Rosenbaum and Ferguson; writing teams, Wylie and Oberle, Copeland and Coffman, Ringenberg and Sitomer; a rewrite team, Good, Levi and Scott; a problems team, Mashin, McCarter, Murphy; a teachers commentary team, Spencer and Brown.

There were many meetings of the entire writing team to settle issues as they arose and to react to preliminary drafts of chapters as they were written. One general session was called for the purpose of giving our geometry book a name. "Geometry With Coordinates" may not be a good title, but be assured that it came into existence as a result of democratic processes. Although the team assignments listed above indicate the primary duties of the individuals of the team, it should be pointed out that we did function as a team. There was a lot of discussion during working hours as well as off-duty hours regarding the subject matter at hand. Some chapters went through four drafts. Each draft was prepared in multiple copies and distributed to all members of the GW team. Each of us knew what the various teams were currently working on. Each of us was expected to react to every draft of everything that was written. Surely the GW team was a group of eager beavers. For there was a lot of reaction and a lot of rewriting.

The preliminary edition was used in selected centers during the school year 1961-62, including schools in Baltimore, New York City, Newtonville, Atlanta, Omaha, Portland, and San Diego. In September, 1961, there was a one day meeting in Chicago to which the GW team and teachers were invited. The day was spent in discussing the content and philosophy of GW. As the year progressed chapter and quarterly reports were submitted to SMSG by the teachers who were using GW as a text in their classes.

San Francisco. In April, 1962, while the National Mathematics meetings were in session, the following persons met as a steering committee to consider several SMSG matters: Dr. Begle, Mr. Ferguson, Professor Ringenberg, Professor Rosenbaum, Mr. Sitomer, Professor Spencer, and Professor Wylie. Dr. Begle reported that the SMSG Advisory Board did not wish, at that time, to accept a recommendation from the Steering Committee that a sample textbook be prepared stressing affine geometry based on coordinates. At the same time, he reported that the Board found the project an interesting one and would be happy to provide assistance to any group working in this area.

A suggestion that Ringenberg, Sitomer, and Spencer meet with Professor Good in Baltimore in June to plan for the 1962 writing session was approved.

It was with deep regret that we learned of Mr. James Brown's death just a few days before our meeting in San Francisco. Jim had served as a member of the Yale GW team and had planned to be at Stanford with his family for the summer 1962 writing session.

Baltimore. Early in June, 1962, Good, Ringenberg, Sitomer, and Spencer met in Baltimore to plan for the 1962 summer writing session. We spent a profitable afternoon and evening discussing with a group of the teachers in the Baltimore center their experiences with GW. The remainder of our three-day session was spent in studying evaluations by the teachers and in planning for the eight week writing session to start at Stanford late in June.

Stanford. The Stanford GW team consisted of the following persons: Coffman, Good, Murphy, Ringenberg, Scott, Sitomer, Spencer, and several high school teachers who joined the group for the first time: Mr. James Hood, San Jose, California; Mr. Michael Joyce, New York City; and Mrs. Dale Rains, Baltimore, Maryland.

Our method of working was similar to that which we had used at Yale. Good and Sitomer, writing more or less independently, wrote first drafts of major revisions of several of the early chapters of the book. Mrs. Scott served as chief of the Problems Section. Spencer and Murphy were assigned to the Teachers Commentary. Miss Coffman wrote revisions of portions of several chapters and helped with the problems. Hood, Joyce, and Mrs. Rains worked on problems revisions and along with the rest of the team reacted to the various rewrite drafts as they appeared. Ringenberg served as chairman. Joyce assisted him as leg man and assistant team coordinator.

II. PHILOSOPHY

Rigor. Geometry With Coordinates is a text in formal geometry for secondary school students. It is not a treatise, but it rests on a sound basis. Its postulates and its sequence of theorems could be rewritten as a treatise to satisfy the mathematically sophisticated without rebuilding the logical structure. The level of rigor is appropriate for "college-capable" high school youth.

Incidence and Existence. In GW, geometry is developed as a theory of points and of various sets of points. The language of sets is developed through examples. Point, line, plane, set, and element of a set are undefined notions in GW. The formal structure begins with a definition (space is the set of all points) and nine incidence and existence postulates. In the Preliminary Edition the first few postulates do not ensure the existence of any points in space. In the Revised Edition the incidence postulates are designed to prevent a difficult pedagogical situation which might arise if there were questions of existence, questions about proofs concerning things which might not exist.

Postulate 1 tells us that space contains at least two distinct points. Postulate 2 tells us that every line is a set of points and contains at least two distinct points. Postulate 3 tells us that, if P and Q are any two distinct points, there is one and only one line containing P and Q . From Postulate 1 we know that space contains at least two points. The hypothesis in Postulate 3 can be satisfied. From Postulates 1, 2, and 3 and the definition of space it is easy to deduce that there is at least one line in space.

Postulate 4 tells us that no line contains all points of space. Postulate 5 tells us that every plane is a set of points and contains at least three noncollinear points. Postulate 6 tells us that if P, Q, R are any three noncollinear points there is one and only one plane which contains them. From Postulates 4, 5, 6 it is easy to deduce that there is at least one plane in space.

Postulate 7 tells us that no plane contains all points of space. (At this stage of the development we have in fact introduced "solid" geometry.) Postulate 8 tells us that if two points of a line belong to a plane, then the line is a subset of the plane. Postulate 9 tells us that if the intersection of two distinct planes is not empty, then it is a line.

At this stage of the development we can prove that space contains four noncoplanar points. But we cannot prove that space contains more than four points. Indeed, one of the problems at the end of Chapter 2 requires the student to show that a certain space of four points does satisfy the nine postulates.

These nine incidence postulates set the flavor for the first part of the course. The postulates are abstract precise statements couched in set language. They are suggested by our experience with physical objects. They are clearly stated; there are no hidden meanings.

The proofs of theorems at this stage are informal paragraph proofs. The students get into the act by occasionally identifying the postulates or preceding theorems which support the assertions in our deductions.

Distance, *measure*, *coordinates*. The possibility that space may contain only four points implies that we need more postulates. We get an infinite number of points on every line via the Ruler Postulate. But first we introduce distance.

A distance is a combination of a number and a unit of distance. Different men, in different times and places, use different units, and so it seems important to consider the effect of different units in establishing a postulational basis for geometry.

In GW any pair of distinct points may serve as the unit of distance. According to our first distance postulate there is, for every pair of distinct points A and A', a distance function which associates with every pair of distinct points P and Q some positive number called the distance between P and Q relative to $\{A, A'\}$ and denoted by PQ(relative to $\{A,A'\}$), sometimes briefly by PQ. We define the distance between any point and itself to be zero. Another postulate makes the distance behave like it should in the sense that if $\{A,A'\}$ and $\{B,B'\}$ are any two unit-pairs, then for all pairs of distinct points P and Q the number

$$\frac{PQ(\text{relative to } \{A,A'\})}{PQ(\text{relative to } \{B,B'\})} \text{ is a constant.}$$

A coordinate system on a line relative to a pair $\{A,A'\}$ is a one-to-one correspondence between the set of all real numbers and the set of all points on the line having the property that if r and s are the numbers matched with points R and S then $RS(\text{relative to } \{A,A'\}) = |r - s|$. This is a definition and actually might have no substance for our geometry. For the definition amounts to saying that if there is such a one-to-one correspondence, it may be called a coordinate system. It does not assure the existence of such a thing. We want this existence and we get it via the Ruler Postulate, which states that if $\{A,A'\}$ is any unit-pair, if is any line, if P and Q are any two points on , then there is a *unique* coordinate system in relative to $\{A,A'\}$ in which P is the origin (number matched with P is 0) and the coordinate of Q (meaning the number matched with Q) is a positive number. Early in Chapter 3 the development of the text assumes a flavor of geometry with coordinates. Rays and segments are defined in terms of coordinates. The key theorems in our development of the geometry of the line using coordinates are the Two Coordinate Systems Theorem, which establishes the relationship between any two coordinate systems on a line $(x' = ax + b)$ and the Two Point Theorem, which develops the key equation, $x = x_1 + k(x_2 - x_1)$. In this equation, x_1, x_2, x , with $x_2 \neq x_1$, are the coordinates of points X_1, X_2, X on a line in one coordinate system. The theorem asserts that k is the coordinate of X in the coordinate system which matches 0 with X_1 and 1 with X_2 . It is motivated by a picture which suggests that the distance of X from 0 is the distance of X_1 from 0 plus a multiple of the distance from X_1 to X_2 . The motivation may be made using the order arrangement for points suggested in the figure. The proof, of course, rests on the postulates and applies to all possible order arrangements. Using the set builder symbol we may write = $\{X: x = x_1 + k(x_2 - x_1)$, k is real$\}$.

In GW the concept of betweenness for points on a line is based on the betweenness relations of the coordinates of the points in some coordinate system. Of course we prove that if B is between A and C using one coordinate system, then B is between A and C if we use any other coordinate system. The statement, "B is between A and C implies $AB + BC = AC$," is a theorem in GW. Thus betweenness for points is based on the order properties of real numbers. Rays and segments are defined using inequalities. Referring to the figure in which we assume $x_1 < x_2$, the segment with endpoints P and Q , denoted by $\overline{PQ}$, is the set of all points X such that $x_1 \leq x \leq x_2$; the ray with endpoint P and containing Q , denoted by $\overrightarrow{PQ}$, is the set of all points X such that $x \geq x_1$.

A typical line coordinate geometry exercise is as follows. If the coordinates of P and Q are 5 and -7 respectively, find the coordinate x of the point X on PQ such that X is in the ray opposite to PQ and $PX = 7 \cdot PQ$. Solution: $x = 5 - 7(-7 - 5) = 89$. Another typical exercise is to find the coordinates of the trisection points of segment $\overline{AB}$ if the coordinates of A and B are 5 and 29. Solution: $x_1 = 5 + \frac{1}{3}(29 - 5) = 13$, $x_2 = 5 + \frac{2}{3}(29 - 5) = 21$.

Just as we introduce distance in Chapter 3 with a discussion of different units of measure, so also we introduce angle measure in Chapter 4 with a discussion of different units. A postulational basis for angle measure with different units could have been given. We decided not to do it because there is so much similarity with the postulational development of distance, and we felt that the usual time allotted for the geometry course does not permit it. Our postulational development of angle measure begins with a postulate which, in effect, assumes the existence of a degree measure for angles. It states that there is a correspondence which associates with each angle in space a unique number between 0 and 180 .

As the angle measure postulate suggests, there is no "zero angle" or "straight angle" or "reflex angle" in GW. An angle is defined as the union of two distinct, noncollinear, concurrent rays. This definition is adequate for GW geometry and simplifies the development. Corresponding to a coordinate system on a line, we introduce a ray coordinate system in a plane. Such a system matches the set of all rays in the given plane and having a given point as endpoint in a one-to-one manner with the set of all real numbers between 0 and 360 , 0 included, 360 excluded. Let R and S be two distinct rays in this system with respective coordinates r and s . If R and S are noncollinear, then the angle which is the union of R and S has measure $|r - s|$ or $360 - |r - s|$; $|r - s| = 180$ if and only if R and S are opposite rays. Corresponding to the Ruler Postulate there is the

Protractor Postulate which spells out the details of the existence and uniqueness properties of ray-coordinate systems.

Separation. The foundation of GW includes two separation postulates, each involving the modern notion of a convex set. The plane separation postulate says: For any plane and any line in that plane, the points of the plane not contained in that line form two disjoint convex sets having the property that every segment which joins a point of one of these convex sets to a point in the other convex set also intersects the given line. Similarly, the space separation postulate tells us that every plane separates space. This development of separation provides a basis for the concepts of the interior and exterior of an angle and of a triangle. The interior of an angle could be defined in terms of two halfplanes which are naturally associated with the angle, or in terms of a set of rays associated with the angle, or in terms of a set of segments associated with the angle. The GW version of geometry incorporates all of these ideas in its Interior of an Angle Postulate. Given any angle $\angle AVB$, let R be the union of the interiors of all rays VC which are between VA and VB, let I be the intersection of the halfplane with edge VA and containing B and the halfplane with edge VB and containing A, let S be the union of the interiors of all segments joining an interior point of VA to an interior point of VB. Then the postulate tells us that $R = I$ and that $R \quad S$. According to GW philosophy the best way to get these facts into our elementary geometry course is to postulate them. Why not go a step further and assume that $R = S$, which of course it is in Euclidean geometry? It is not assumed because (1) it is not needed at this point in the development, and (2) $R = S$ would imply the famous parallel postulate which appears several chapters later in the book.

Congruence. In GW the concept of congruence rests upon the concept of measure and is developed for segments, angles, triangles, convex polygons, circles and arcs. Two segments are congruent if they have the same length. Two angles are congruent if they have the same measure. Triangle congruence in GW is a property of a correspondence between sets of vertices. Given the two triangles ABC and DEF, correspondence $ABC \quad DEF$ which matches A with D, B with E, C with F, is a congruence, and we write $ABC \cong DEF$, if the parts which are matched under the correspondence are congruent. In other words, $ABC \cong DEF$ if and only if $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$, $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, $\overline{BC} \cong \overline{EF}$. The statement that $ABC \cong DEF$ does not imply that ABC and DEF are different triangles. $ABC \cong ABC$ is true for every triangle ABC; $ABC \cong ACB$ is true if the six definitional requirements are all satisfied.

In GW the familiar triangle congruence "theorems" of traditional geometry are motivated by means of experiments with cardboard triangles. Then all three of them, abbreviated by S.A.S. Postulate, A.S.A. Postulate, S.S.S. Postulate are postulated. The judgment of the GW team was that they should be postulated as soon as their meaning and significance are clear to the students. Exploiting these postulates in deducing theorems about triangles occupies a large portion of Chapter 5. It is in this Chapter that the process of creating a two-column proof is carefully developed. This development includes a discussion of the nature of a deductive proof and an explanation of how number principles are used in proofs about geometric objects.

Parallelism and Similarity. In the revised edition of GW we define lines and to be parallel if and are coplanar and the intersection of and is not a set consisting of one point. This implies that every line is parallel to itself. It was the judgment of the GW team that the advantages of this departure from the traditional definition outweigh its disadvantages. In order to complete the discussion as regards incidence relations for lines, and in view of the fact that the geometry of a plane is developed as a part of the geometry of space, the concept of skew lines is introduced along with the concept of parallel lines in the beginning of the chapter on parallelism. The definitions of transversal, alternate interior angles, consecutive interior angles, and corresponding angles are expressed with a modern flavor in terms of intersections and unions.

The Parallel Postulate is introduced with some motivating remarks which include a short discussion of non-Euclidean geometry. The purpose of these remarks is to show the historical importance of the Parallel Postulate and to give the student a better understanding and appreciation of the postulational method. The GW development makes it clear that certain important theorems do not depend upon the Parallel Postulate. These are the theorems of neutral geometry and include the isosceles triangle theorem and its converse; the exterior angle theorem (the measure of an exterior angle of a triangle is greater than the measure of either interior angle not adjacent to it); the theorem which states that if a transversal of two coplanar lines determines with them a pair of congruent alternate interior angles, then the two lines are parallel; and the theorem which states that, given a line and a point not on the line there is at least one line through the point and parallel to the given line. It is emphasized that without the Parallel Postulate it is impossible to prove that there is no more than one line through a given point and parallel to a given line. In the chapter on parallelism we include a number of theorems based on the Parallel Postulate and relating to parallel lines, parallelograms, and the sum of the measures of the angles of a triangle. Other theorems relating to these topics and found in many of the standard textbooks are deferred until one of the later chapters in which coordinates or vectors are used. We also defer until a later chapter the general theorems concerning the sum of the measures of the interior and the exterior angles of any convex polygon. This permits us to get to the Pythagorean Theorem more quickly; and this we desire so that we may introduce coordinates in a plane and in space.

The GW treatment of similarity is somewhat different from the usual ratio approach. We define the numbers $q, r, s, \ldots$ to be *proportional* to the numbers $a, b, c, \ldots$ if there is a non-zero constant k such that $q = ka$, $r = kb$, $s = kc$, ... The constant k is called the constant of proportionality. In the revised edition we introduced the notation of the following example: $(6,12,21) \underset{p}{=} (2,4,7)$ means that 6, 12, 21 are proportional to 2, 4, 7 .

This treatment of proportionality is more general than the usual treatment, and we believe it is simpler. It is our purpose to use a definition which avoids stating exceptional cases involving zero in the applications of the theorems on similarity to the development of coordinate geometry which follows. This general approach should be useful to students in their study of more advanced mathematics.

The properties of proportionalities are followed by a treatment of similarity for polygons. Congruence for polygons is the special case of similarity which results when the constant of proportionality is 1. The GW treatment includes a Proportional Segments Postulate: If a line is parallel to one side of a triangle and intersects the other two sides in interior points, then the measures of one of those sides and of the two segments into which it is cut are proportional to the measures of the other side and the corresponding segments thereof. We adopted this statement as a postulate because we feel it is intuitively reasonable and because it avoids many difficulties inherent in proving the statement for the incommensurable case.

The Proportional Segments Postulate is used to prove the standard theorem concerning three parallel lines and two transversals and the proportionality involving the measure of the segments of the transversals. The triangle similarity theorems (three of them corresponding to the three triangle congruence postulates) are motivated as being useful statements which reduce the amount of work required by the definition in showing that triangles are similar. The similarity chapter reaches its climax with the Pythagorean Theorem.

An idea which occurs repeatedly in GW is the concept of an equivalence relation. The properties of such a relation, properly named, appear in the GW discussions of congruence for segments, congruence for angles, congruence for triangles, similarity properties, and proportionality properties.

Coordinates in a plane and in space; space geometry. Coordinates are introduced in GW as a tool for studying geometry. The development in Chapter 8, Coordinates in a Plane, includes a sequence of basic theorems, the distance formula, midpoint formula, parametric equations for a line, a treatment of the slope concept, perpendicularity and parallelism conditions, and the use of coordinates in proving several theorems about triangles and quadrilaterals.

The GW treatment avoids a contest between synthetic and coordinate geometry. We hope the students do not get the idea that synthetic geometry and coordinate geometry are two kinds of geometry. We prefer that they see synthetic and coordinate methods as two methods for studying the same geometry. We prefer not to think of our treatment as an introduction to analytic geometry. The traditional anlytic geometry course includes various standard forms of equations for lines and conic sections. It emphasizes the plotting of graphs and the find of equations of curves from information given about the graphs. It places little emphasis upon the use of coordinates in the formal development of the elementary geometry of lines, triangles, and quadrilaterals. Why should it? The students have this background before they enter the course.

The GW treatment emphasizes the use of coordinates by using them in the development of several theorems. The student sees a sequence of theorems in which some are proved using coordinates and some are proved without using coordinates. The student is supposed to get the idea that coordinates are used if it makes things easier and that coordinates are not used if it is easier that way. The student who has studied coordinates on a line, coordinates in a plane, and coordinates in space, achieves a depth of understanding of basic geometry which it is impossible to get with synthetic methods alone.

Corresponding to the key theorems regarding coordinate systems on a line suggested by the equation $x = x_1 + k(x_2 - x_1)$ and

the above figure are the following two theorems.

Theorem. If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are any two distinct points, then

$$P_1P_2 = \{(x,y): \ x = x_1 + k(x_2 - x_1) \ , \ y = y_1 + k(y_2 - y_1) \ , \ k \text{ is real}\} \ .$$

Theorem. If a, b, c, d are any real numbers with b and d not both zero, and if

$S = \{(x,y):\ x = a + bk\ ,\ y = c + dk\ ,\ k \text{ is real}\}$ then S is a line.

These set builder symbols with their parametric equations emphasize the fact that a line is a set of points. Referring to the first of the two theorems, suppose a value is assigned to k. It is easy for the student to "see" the resulting point (x,y) in relation to the points (x_1,y_1) and (x_2,y_2). The student does not think of the parametric equations as statements of properties of the coordinates. Rather he looks at them together as "generating" the points of a line.

We use coordinates in space in developing parametric equations for a line P_1P_2 in terms of the coordinates of the points P_1 and P_2. Our treatment of space geometry makes no pretense at completeness. The GW philosophy is in accord with the Commission Report that there is neither time nor virtue in presenting a complete formal treatment of space geometry in the first year's work in formal geometry.

The objective of the GW treatment of space geometry is to get the students to see and to discover spatial relationships. We have used an intuitive approach in the form of exploratory problems. We have proved some of the theorems in order to show the student that there is nothing peculiar about proofs of theorems in three-dimensional geometry. We have left many proofs for the problems and expect that most teachers will assign very few of these problems. We have omitted other proofs completely.

Vectors. Chapter 10, Directed Segments and Vectors, is an optional chapter and should be omitted if the ability level of the class or the lack of time makes omission of some chapter necessary. The purpose of this chapter is to introduce the above average student to another useful tool for the study of geometry, a tool which has wide applications in engineering and physics as well as in mathematics.

The chapter begins with a formal development of the geometry of directed segments. This development includes a definition of the equivalence of directed segments in terms of their components followed by a sequence of theorms regarding equivalent directed segments.

A vector is defined as a pair of components. If P and Q are distinct points, (P,Q) denotes the directed segment from P to Q while PQ denotes the vector whose components are the same as the components of (P,Q) . The effect of this treatment is that a vector is a whole set (actually an equivalence class) of directed segments, and that any one of these directed segments may "represent" the vector. This is in accord with modern approaches to vector analysis and should make it easier for the student to make the transition from this course in elementary geometry to a study of vector analysis.

The development of the work in vectors includes several theorems regarding parallel and perpendicular vectors, the addition and subtraction of vectors, and the theorem about expressing one vector as a linear combination of other vectors. Vectors are used as a tool in the development of formal geometry; vector proofs are given for well-known theorems (1) one regarding the figure which is formed by joining the midpoints of the sides of a quadrilateral, (2) one regarding the line segment joining the midpoints of two sides of a triangle, and (3) one which states that a quadrilateral is a parallelogram if and only if its diagonals bisect each other. Several of the problems in the chapter on vectors are problems in physics. The only physics which a student need know in order to work these problems is that when two or more forces or velocities act at a point on a body the resultant force or velocity can be found by the rules of vector addition. Although the chapter deals almost exclusively with coplanar vectors, the viewpoint admits immediate extension to vectors in three or more dimensions.

Polygons, polyhedrons, circles, spheres. Polygons and polyhedrons are treated in Chapter 11, circles and spheres in Chapter 12. The work on polygons and polyhedrons is conventional subject matter treated essentially from Euclid's viewpoint with proofs similar to those in many of the standard textbooks. One difference in the GW treatment is the introduction of the term polygonal-region. Another is the study of area by postulating the properties of area rather than by deriving the properties from a definition of area based on the measurement process. Actually both of these approaches are implicit in the conventional treatment. The GW treatment brings them to the surface and clarifies them.

Earlier in the text polygon, convex polygon, and interior of convex polygon had been defined. We define a triangular-region as the union of a triangle and its interior. (Note that a triangle is a convex polygon and its interior has been defined.) We avoid defining the interior of an arbitrary polygon by defining a polygonal-region in terms of triangular regions.

The work on circles and spheres includes the conventional subject matter. Coordinates are used in the proofs of some theorems, e.g., the theorem regarding the intersection of a plane and a sphere, and the theorem regarding the intersection of a circle and a line in the plane of the circle.

The circumference of a circle in GW is a real number. It is defined as the limit of the perimeters of the inscribed regular polygons as the number of sides increases. The geometry here is well motivated. The formal geometry is complete; but the formal development of the limit concept is, of course, omitted. The area of a circle is defined as the limit of the area of the inscribed regular polygons. Again the development is complete except for some of the preoperties of limits which are hidden in some pretty arrows.

The last two chapters include six new postulates, four in Chapter 11 regarding areas and two in Chapter 12 regarding lengths of arcs of circles and their degree measures.

CRITICISM OF SMSG TEXT MATERIAL

N. E. Steenrod

A serious error has been made which involves nearly all of the text material from the seventh to the twelfth grades. I feel quite strongly that this must be rectified. Here is a brief statement of the indictment:

The concept of function is not mentioned until the middle of the 11th grade text, and is not presented in full generality until the 12th grade. The result of this omission is unnecessary confusion, some nonsense, and a failure to give to the mathematics the kind of unity it should and does have.

It is generally agreed that the language of modern mathematics is the language of sets and functions. All of the SMSG texts use "sets" and related concepts in a most satisfying way. But "functions" are avoided. Instead, the words correspondence, operation, variable, and open sentence are used; but it is not said anywhere that these are all examples of the same thing. Why should we give the students half a loaf when the complete meal can be provided at no extra cost, and is more digestible?

In the 7th and 8th grade texts, the word "correspondence" is introduced but is used rather rarely. Although functions appear on every page, this fact is never mentioned. In the material on numbers and arithmetic, addition, multiplication and division are called "operations." In the algebra texts, the letters of the alphabet sometimes denote "variables" and sometimes "numbers." Correspondences are used again and very frequently in the 10th grade geometry texts. In the 11th grade algebra, a "function" is defined to be a correspondence from a set of numbers to another set of numbers. Not until the 12th grade text is a proper definition given of function.

And then its connection with operation and variable is not mentioned. The word "association" is used in this definition; and, on the preceding page, the expression "association or correspondence" occurs. But it is not spelled out that the correspondences of geometry and the variables of algebra are examples of functions.

Thus the student must learn to use different words for each "compartment" of mathematics: operations in arithmetic, correspondences in geometry, variables in algebra, and functions in advanced algebra. Nowhere is it stated that these are names for the same concept appearing in slightly different contexts. And yet a major objective of our new program is to reduce the compartmentalization and increase the unity. In this instance, the present texts have failed.

The worst example of the consequences of this failure appears in the explanation of "variable" on pages 38-41 of the 9th grade algebra text.* The text gives only an informal definition of variable roughly as follows. Each student in the class picks a number. They proceed to play a game involving John's number (known only to John). The student, Don, decides that the expression "John's number" is too long to write, so he abbreviates it by "n"; and he arrives at the conclusion that

$$2(n + 3) - 2n = 6$$

no matter what John's number is. At this point there are two paragraphs as follows:

> "Notice that we have used the letter "n" (Don's abbreviation of "John's number") as referring to a particular number, but as far as we know, this number may be any one of the set S. Would Don's reasoning have changed if he had called the number "t"? "a"? "k"? Are there any restrictions on his choice of a symbol to designate a number from S?
>
> "We call a letter such as "n," when used as Don

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* First Course in Algebra (Preliminary Edition)

> used it, a *variable*. The numbers that n may be are called its values."

I submit that the foregoing is a lovely mixture of confusion, nonsense and falsehood. An intelligent student would be justified in concluding that a variable is a number known only to John, and the reason for the name "variable" is that we can denote John's number by any of the various letters of the alphabet.

The falsehood becomes obvious as soon as the situation under discussion is analyzed in terms of sets and functions. Each student chooses a number from the set $S = (1, 2, \ldots, 30)$. This defines a function $f: D \longrightarrow S$ where D denotes the set of students. Then f(John) is a number, it is a value of the function, and it is not a variable. The variable is the function f.

There is no mystery in the foundations of algebra if we take the attitude that the concept of function is just as simple and primitive as that of set. The definition of algebra can then be given as follows: Let D denote a set (unrestricted), and let R denote the set of real numbers. The totality A of functions with domain D and range R forms an algebra under the addition and multiplication of functions. Thus: *algebra is the arithmetic of real-valued functions*. The set D is specified in different ways in different applications. For example, in the formula for the area of a circle $a = \pi r^2$, the set D is the set of circles, and a, r are real-valued functions defined on D.

On page 29 of the Commentary for the first course in algebra appears another fine example of the kind of nonsense that arises when functions are avoided:

> "Nevertheless, x actually *is* the name of a number, it's not just holding the place for a name of a number. ... how can $x + 3$ mean anything unless x is the name of a number? You can't add a number to a place you are holding for a number!"

I could not find a definition of "place holding". However, if x denotes a real-valued function on a set D, and 3 denotes the constant function defined on D, then $x + 3$ is a perfectly good function.

To rectify the general mistake that has been made should not be difficult. Minor changes in each of the texts would do the job. The following paragraphs suggest how this can be done.

In the 7th grade text, the word "function" should be introduced and used without fuss. This might best be done near the middle, and after the concept of set has taken hold. It should be introduced and used along with "correspondence" as a synonym.

Near the beginning of the 8th grade text, the formal definition of function with domain and range can be given. Several lessons should be devoted to teaching the student how to recognize functions and determine their domains and ranges. Many examples from everyday life can be given, viz: the mother of a boy, the teacher of a class, the color of a book. The student should learn that whenever the word "of" is used or whenever the possessive form of a noun appears, then there is a function in the offing. He should also be taught to distinguish between the function and a value of the function, e.g. the vicar of a parish, and the Vicar of Wakefield. Throughout the remainder of the 8th grade text, the student's attention should be called to the numerous functions which appear. In particular the operations of arithmetic are functions.

In the 9th grade text, the definition of algebra would be given in terms of real-valued functions defined on a set (as described above). The word "variable" might well be introduced here to mean a function whose range is the set of real numbers.

In the 10th grade geometry, it need only be stated that distance, correspondence, congruence, similarity and bisector are all examples of functions.

In the 11th grade algebra, the definition of algebra should be restated and emphasized at the beginning together with the meaning of variable as a special kind of function. The present section in which a function is defined as one whose domain is a set of real numbers should be dropped. One can proceed directly to the graphs of functions from real numbers to real numbers.

Only minor and obvious modifications are needed in the 12th grade texts.

REVIEW OF
THE
SMSG FIRST COURSE IN ALGEBRA*

by

Peter D. Lax

The pilot highschool texts produced by SMSG have exercised a considerable influence on the recent crop of commercially produced texts, and will continue to do so for some years to come. Therefore it is important to examine these pilot texts critically and point out what is good in them and what is not so good.

This review examines the most important volume of the series, the First Course in Algebra, scrutinizing its general philosophy and the details of the execution.

The book has a number of attractive new features:

1. The inclusion and careful discussion of somewhat sophisticated topics, such as the irrationality of $\sqrt{2}$, elementary facts about primes and the Newton iteration method for extracting square roots.

2. The quality of the exercises - they are not mere routine drill exercises but confront the student with unexpected situations and make him think.

3. The careful explanation of the logic behind transforming equations and inequalities into equivalent ones in the process of solving them.

4. A novel approach in training students to translate worded problems into the language of algebra.

5. The discussion of inequalities as well as equalities.

6. The absence of fancy terminology and symbolism.

Like all human products, these volumes have imperfections, some no doubt due to the circumstance that they were written in a great hurry and by a committee. The shortcomings of this text may be put in these four categories.

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* SMSG, Yale Univ. Press, 1960, Part I, Part II.

1. Lack of motivation
2. Logical errors
3. Mathematical errors
4. Heavy-handed and pedantic exposition.

Since quite a lot has been written about the virtues of new math., I shall take the role of the devil's advocate and discuss the shortcomings:

1. Lack of Motivation: The average intelligent highschool student is entitled to an honest explanation what a subject is about, in terms that are meaningful to him. The way to do this in mathematics is to present challenging problems, and then to show the student how the theory developed helps to solve these problems. The present volume fails to do this or emphasize this in a number of places, most notably in the first part where negative numbers and operations with them are introduced. The authors assert repeatedly (see e.g. page 145) that "it is a primary importance to preserve the structure of the number system", without saying why it is important. This is strange since it is possible to give an explanation which is meaningful on the highschool level:

"In solving equations or inequalities we do not know beforehand whether the unknown will turn out to be a positive or negative number. Therefore it is indispensable to have uniform rules of manipulation."

I don't like it that the authors are content to rest the case for $(-1) \times (-1) = 1$ solely on the distributive law. They should also point out that this definition is suggested in some important applications, such as multiplying negative time (in the sense of referring to events which took place prior to an arbitrarily fixed reference time) by negative velocity. This is important for two reasons: it shows that the rule promulgated is useful in applications, and it shows how an application itself suggests the framing of a mathematical definition.

I object to the authors' introductory statement to Chapter 15, "Let us consider compound sentences of two clauses, etc." because they don't say why we are moved to consider such objects. The true reason is again simple to state and easy to comprehend:

"In many problems we are trying to determine one or several numbers about which we know two or more pieces of information stated as equations and inequalities."

This should be followed by examples from everyday life, such as the ones the authors themselves give in the exercises on page 484.

The authors are on the right track in Chapter 14, where they introduce analytical geometry by saying that, since the real number line has been useful, a real number plane would be perhaps even more useful. I would have added here that its usefulness lies in enabling us to use our geometric intuition to solve analytic problems. (This chapter and the next contain a great variety of interesting problems.)

Chapter 12 starts out similarly on the right track, by observing that since factoring numbers turned out to be advantageous, factoring algebraic expressions might be similarly useful. Unfortunately this chapter is marred by an overemphasis of the traditional method of factoring. On top of page 333 the authors say casually that some polynomials can be factored by the method of perfect squares and difference of squares! They should be saying instead that all quadratic polynomials can be factored by this method, more precisely the method will tell you when a quadratic polynomial can be factored in a given field. So why feature the hunt-and-peck method? Is it just adherence to tradition?

The authors blunder on page 348 when they say that "since a polynomial over the rationals can be written as a product of a rational number and of a polynomial over the integers, the problem of factoring polynomials over the rationals is reduced to the problem of factoring over the integers". It isn't so; just try it on the polynomial $x^2 - 2$, discussed in that same section and see if you can prove its irreducibility over the rationals. (Incidentally, the irreducibility of this polynomial is equivalent to the irrationality of $\sqrt{2}$, proved elsewhere impeccably.)

2. *Logical errors*. There is a serious logical error in Chapter 4 in the treatment of open sentences. The authors start with the good idea to regard writing equations for worded problems as translation from English to Algebra, and proceed to give the student a lot of practice translating back and forth. Unfortunately, the authors don't realize that whether a sentence is open or not *does not depend on the language in which it is expressed*. Thus an open algebraic sentence must be translated as an *open* English sentence. The authors seem to be aware of this on page 42 where they quote "he is a doctor" as an open English sentence, but then they forget about it. E.g. on page 46 they assert that many formulas used in business and science are open sentences. If they were, they would be useless!

I have always felt that open sentence is an artificial concept at the highschool level. That the authors of this volume stumbled over it confirms my conviction.

On pages 132 and 165 the addition and multiplication properties of equality are introduced. This is mighty strange in a curriculum which emphasizes so strongly the number concept, goes to such lengths not to confuse it with the various names for numbers, and emphasizes the concept of operation with numbers. You can't have it both ways! What is worse, the authors state on page 132, line 3^*, that this principle is *another* fact about addition; the discussion following it suggests that the evidence for this fact is inductive.

The properties of equality listed on page 205 are similarly unnecessary.

I am distressed that most modern texts feature one form or another of these unnecessary principles or properties of equality. This debases the role of logic to pedantic parrotting of phrases, without which the homework is marked wrong. After all, when the student concludes that the President's hat is the same as Lyndon Johnson's hat, he needs no special hat principle of equality, merely the basic notion of *sameness*. It is no different in mathematics.

3. Mathematical errors. The most disturbing mathematical error is the omission of the subtraction property* of positive numbers from the list on page 71 among the basic properties of the numbers of arithmetic. The authors use it, as they must, to define addition of positive and negative numbers (page 127); why did they fail to list it?

In Chapter 8 the authors for the same reason are unable to make up their minds whether order is a defined or undefined concept. On page 190, line 18, they use quite casually the fact that $0 < b$ means the same as saying that b is positive. Why isn't this labeled as a basic property? Rather, why not define $a < b$ to mean that $b - a$ is positive?

The omission of the subtraction property makes shambles of the authors' attempts to present rigorous proofs of theorems. E.g. on page 130 they state the associative property of addition for real numbers, but skip the proof, saying that it would be too long. It would be more correct to say that the proof would have to use some form of the subtraction property. The unproved (unprovable) associative law is then used on page 137 to show, at some length, that every real number has exactly one additive inverse!. The result is of course an immediate consequence of the definition.

On page 58 the authors say that equivalent fractions, say $\frac{5}{6}$ and $\frac{15}{18}$, are equal because the latter results from the former by multiplication by $1 = \frac{3}{3}$. This assumes that you know already how to multiply fractions but don't know yet when two fractions represent the same number. This is putting the cart before the horse.

There are two further mathematical errors, minor in the sense that they don't pervade other parts of the book but nevertheless annoying. One is the statement on page 23-24 that it is impossible to add more than two numbers simultaneously. This is a curious error in the light of the recent (over) emphasis on the set concept; also it contradicts the common experience (cited by the authors themselves) of adding columns of figures.

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* If a and b are unequal positive numbers, there exists a unique positive x such that either $a + x = b$ or $b + x = a$.

The authors aggravate their error on page 24 by verifying the associative law on four examples and then saying: "These sentences have a common pattern, and they are all true; therefore every sentence having this pattern is true". This is exceedingly poor logic. As a matter of fact, our belief in the associative (and commutative) law is *not* based on inductive evidence; seeing one more instance of it would in no way strengthen our faith in it. The plausible proof for these laws is based on adding several numbers simultaneously by forming the union of several sets, the possibility of which the authors have denied.

4. *Heavy-handed*, *pedantic exposition*. This criticism does not of course apply to every part of the book. Some sections, notably the beginning of Chapter 6 and of 17, are masterpieces of clarity and of light touch. But the whole first volume, describing the number system, is heavy, pompous, overweight. I perceive two sources of this; one is philosophical: an overemphasis on proofs. After all, the importance of proofs in elementary algebra is for thought economy: the student has to remember only a few basic rules, the rest can be deduced. But it is a mistake to pass off these deductions* as minor intellectual gems; if there is nothing surprising in the conclusion, they are (for most people) dry as dust.

The second source is technical: the choice of the postulates of an ordered field; it is incompatible with developing the real numbers starting from the positive ones.

To sum up: Future textbook writers as well as school administrators and teachers should look upon SMSG material not as a new orthodoxy but in the spirit in which it is offered; a lot of new ideas, carefully worked out, but still in need of much critical evaluation, on the basis of mathematical and pedagogical principles and also classroom experience.

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* Especially if they are as poorly done as those in the first volume.